Journal of Statistical Physics, Vol. 121, Nos. 3/4, November 2005 (© 2005)
DOI: 10.1007/s10955-005-7007-0

Rigorous Proof of Luttinger Liquid Behavior in the 1d Hubbard Model

Vieri Mastropietro[1]

Received February 17, 2005; accepted June 1, 2005

We give the first rigorous (non perturbative) proof of Luttinger liquid behavior in the one dimensional Hubbard model, for small repulsive interaction and values of the density different from half filling. The analysis is based on the combination of multiscale analysis with Ward identities based on a hidden and approximate local chiral gauge invariance. No use is done of exact solutions or special integrability properties of the Hubbard model, and the results can be in fact easily generalized to include non local interactions, magnetic fields or interaction with external potentials.

KEY WORDS: Interacting fermions; spin; Ward identities; Renormalization Group.

1. INTRODUCTION

1.1. Historical Remarks

The Hubbard model describes electrons in a crystalline lattice, hopping from one site of a lattice to another and interacting by a repulsive (Coulomb) force with coupling $U > 0$. Such a model in the theory of interacting electrons has the same role of the Ising model in the problem of spin-spin correlations, that is it is the simplest model displaying many real word features: it is however much more difficult to analyze. It is believed that the Hubbard model gives a correct description of the properties of many metals due to the interactions between conduction electrons: for instance the phenomenon of Mott transition, the anomalous properties of high T_c superconductors or the singular properties of quantum wires. However the

[1]Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, I-00133, Rome, Italy; e-mail: mastropi@mat.uniroma2.it

mathematical complexity of the computations is such that this belief is still far from be substantiated by solid arguments. While our understanding of the Hubbard model in higher dimensions is really poor, the situation is of course better in $d=1$; the interest in such a case is not purely academic as in this case the model is believed to furnish an accurate description of real systems like quantum wires.

In the sixties Lieb and Wu[18] solved the 1d Hubbard model by using a lattice version of the *Bethe ansatz*.[2] It appeared that eigenstates of the 1d Hubbard model can be constructed by solving a complicate set of equations essentially identical to the ones already solved by Gaudin[14] and Yang[26] in the context of fermions with δ-interactions. If only the states of Bethe ansatz form are considered, one sees that the system is insulating at half filling and conducting away from half filling. Not all the states are Bethe ansatz states; however it was shown in ref. 10 that in the half filled band case all the remaining states can be obtained from the Bethe ansatz states using the $SO(4)$ symmetry of the model. Recently Lieb and Wu have reconsidered in ref. 18 their analysis of the 1d Hubbard model and have outlined a strategy for a proof that the lowest energy state of Bethe ansatz form is really the ground state; in particular it was shown that such a property can be deduced from two assumptions: (a) that such a state is a continuous function of the coupling U and (b) that its norm is not identically zero. Such two properties were recently proved in ref. 13 in the half filled band case. Of course many other results were derived by Bethe ansatz solution, see for instance refs. 20 and 25.

While the analysis by Bethe ansatz gives very non trivial informations on the spectrum, it is essentially of no utility for computing the correlations, which are the quantities more directly related to physical observables; even if one has the full form of the wave functions (what is actually *not* the case, as the Bethe ansatz gives them as the solutions of complicate integral equations), computing the correlations from them is essentially impossible. In particular, an important question which cannot be answered by the exact solution is if the Hubbard model is a *Fermi liquid* or a *Luttinger liquid*. The notion of Luttinger liquid was introduced by Haldane[16] in the early eighties. While a Fermi liquid is an interacting fermionic system whose low energy behavior is close to the one of the free Fermi gas, a Luttinger liquid behaves as the *Luttinger model*; a model describing spinless fermions in the continuum with linear dispersion relation and short-range (non local) interaction. The linear dispersion relation has the effect that infinitely many unphysical fermions must be introduced to fill the "Dirac sea" of states with negative energy. This makes the model a bit unrealistic and of no direct applicability to solid state physics but, on the other hand, the choice of a linear dispersion relation has the effect that, contrary to all other models of interacting fermions, the Luttinger model correlations can be explicitly computed, see ref. 19. The popularity of the Luttinger liquid notion increased greatly after Anderson's proposal[1] that the high-T_c superconductors are, in their normal phase, Luttinger liquids; this proposal was based on the conjecture that the Hubbard model in one or two dimensions has a somewhat similar behavior, and in particular that they *both* show Luttinger liquid behavior at least for some range of the parameters. Up to now there is no agreement even at an heuristic level on theoretical evidence of Luttinger liquid behavior in the $d=2$ Hubbard model. On the other hand Anderson's proposal stimulated a number of studies about the Luttinger liquid behavior in $d=1$, as a natural prerequisite to understand the same question in $d=2$.

Numerical simulations of the correlation functions gave evidence[21] that the $d=1$ Hubbard model is indeed a Luttinger liquid; subsequent analytic (but heuristic) results by refs. 22 and 11 confirmed this result for *large values* of U, finding also that the correlations verify an important property, the *spin-charge separation*. For *small* U the evidence for Luttinger liquid behavior is based on the two following facts:

(1) The $d=1$ Hubbard model should be equivalent, as far as low energy property are considered, to a generalization of the Luttinger model to *spinning* fermions, the so called *g-ology* model (with a suitable choice of the couplings).

(2) Contrary to the Luttinger model, even the g-ology model is *not solvable*. However Solyom[24] by Renormalization Group (RG) analysis truncated at two loops showed that the $g-ology$ model scales iterating the RG to the *Mattis model*, a model which is indeed exactly solvable and which shows Luttinger liquid behavior.

Given the above two facts, the formulas for the correlations of the $d=1$ Hubbard model are usually approximated with the formulas for the correlations of the Mattis model, see for instance ref. 24; this is however quite unsatisfactory for a number of reasons.

(1) Assuming the equivalence of the Hubbard with the g-ology model means that the effects of the lattice and the corresponding Umklapp scattering terms in the Hubbard model are completely neglected, as the g-ology model is a continuum model with linear bands. There are however strong indications that this approximation gives completely wrong predictions at least for properties like the thermal or electric conductivity.[23]

(2) The conclusion of ref. 24 that the Hubbard model scales iterating the RG toward the Mattis model is based on a number of peculiar

cancellations in the perturbative expansion, *checked up to two loops*. Of course, without an argument stating that the cancellations are present at *any order*, this conclusion is only perturbative and not very solid; if at higher orders the cancellations were not present, the effective coupling constants could increase without limit making the analysis meaningless.

As a conclusion, the enormous number of results on the $d = 1$ Hubbard model can be roughly divided in two main classes. A first class are the exact results, based on the Bethe ansatz approach. They are (essentially) rigorous but they give no informations on the behavior of the correlations. Moreover they are not very robust, as they rely on delicate integrability properties of the Hubbard model and cannot be used to face apparently harmless modifications of the Hubbard model (for instance considering nonlocal but short ranged interactions). The second class of results is obtained by a combination of techniques (numerical simulations, bosonization, Renormalization Group) and indeed they give informations on the correlations, but these results are not rigorous.

1.2. The Luttinger Liquid Construction

In a series of paper[4–7] a general proof of Luttinger liquid behavior for *spinless* interacting fermions (without any use of exact solutions) has been completed. The conclusion is that interacting *spinless* fermions are *generically* Luttinger liquids (independently from the dispersion relation, the presence of a lattice, the sign of the interaction, etc). A perturbation theory based on Renormalization Group ideas is constructed, and the correlations are written as a series not in the strength of the interaction but in terms of a set of parameters called *running coupling constants*, describing the effective interaction at a certain momentum scale; the expansion is proved to be convergent (and analytic) if the running coupling constants are small, see ref. 4, as a consequence of suitable determinant bounds for the fermionic truncated expectations. On the other hand, the property that the running coupling constants remain in the convergence radius of the expansion is not trivial at all and is due to remarkable *cancellations* at any order in the expansion. More exactly, the running coupling constants verify a set of recursive equations, whose l.h.s. is called *Beta function*, and their boundedness is a consequence of dramatic cancellations happening at any order in the Beta function. In order to prove such cancellations one decomposes the Beta function in the sum of two terms; one, called *dominant part*, which is common to all spinless $d = 1$ Fermi system, and the second part which depends on the specific model and which gives a bounded flow *once one has proved that the dominant part is asymptotically vanishing*. The problem is then reduced to the vanishing of the dominant part of the Beta function,which coincides with the complete Beta function of a model, called *reference model*, describing interacting spinless fermions with an ultraviolet and infrared cutoff. The proof of vanishing of the Beta function is then reduced to the proof of suitable (highly non trivial) Ward Identities between the correlation functions of the reference model,[6] for any value of the infrared cutoff. The problem of implementing Ward identities in a model with cutoffs (and in a Renormalization Group scheme) is a well known problem in Quantum field theory or condensed matter physics. In refs. 5 and 7, a solution for this problem was given by finding suitable *Correction Identities* relating the corrections to the Ward Identities due to cutoffs to the correlations themselves. By combining the Ward and the Correction Identities the vanishing of the reference model Beta function is proved and the rigorous construction of the correlation functions for spinless Luttinger liquid is then completed.

Aim of this paper is the extension of the Luttinger liquid construction to the $d = 1$ Hubbard model; such extension is not straightforward at all as the Luttinger liquid is *not* the generic state of *spinning* interacting fermions. The conditions of repulsive interaction $U > 0$ and not half filling must be imposed; technically this is reflected from the fact that the expansions we find cannot be analytic in a circle around $U = 0$, as it would be in the spinless case, and U must be chosen smaller and smaller as we are closer and closer to half-filling. We will define an expansion for the correlations in terms of running coupling constants, but the presence of the spin increases greatly their number; crucial symmetry considerations (based on the $SU(2)$ spin invariance of the Hubbard model) and geometrical constraints reduce the number of the effective interactions (which is one in the spinless case) to three (in the not half-filled band case) or four (in the half filled band case). Again the running coupling constants verify a recursive relation, whose r.h.s. is called Beta function, and the expansion is meaningful only if the running coupling constant are small at any momentum scale. One can decompose the Beta function in a dominant part and a rest; it turns out however that the dominant part is not vanishing. Calling the three (in the not half filled case) effective interactions g_1, g_2, g_4, it turns out that, truncating the beta function at the second order, g_1 tends to vanish (if $U > 0$) while g_2, g_4 remains close to their initial value. In order to prove that such a result is valid non perturbatively, that is including all orders contributions, one has to prove a property which we will call *partial vanishing* of the Beta function. Such property is derived by a suitable *reference model*, which verifies formally (if cutoffs are neglected) proper gauge symmetries. Quite surprisingly, the cancellations on Beta function of the Hubbard model, which verifies an $SU(2)$ spin symmetry, will be obtained by a reference model *not* $SU(2)$ *invariant*. We derive suitable Ward and Correction Identities for the reference model, and by them the partial vanishing of the Hubbard Beta function is proved. Hence the running coupling constants are small if $U$is small enough and the rigorous construction of the correlation functions for Hubbard model is completed. The analysis can be easily generalized to include the presence of a magnetic field or non local interactions.

Finally, we stress that the computation of the correlations in the cases left out by the present work, that is in the half-filled band or in the attractive case, is a very complex and important open problem. In such cases the running coupling constants tend to increase iterating the Renormalization Group and convergence of the renormalized expansion is obtained only up to exponentially small temperatures; this is probably related to the fact that from the Bethe ansatz analysis the system is an insulating at half filling. Similar problems appear in $d > 1$ Hubbard model; one can still define by Renormalization Group methods an expansion for the correlations in terms of the running coupling constants (indeed they are functions in $d > 1$), but they tend to increase iterating the Renormalization Group, as a consequence of the various quantum instabilities (like superconductivity) which are expected at low temperatures. Hence rigorous results for the correlations in interacting Hubbard-like models in $d > 1$ are at the moment found only when instabilities are absent; this can be obtained or considering large enough temperatures, like in refs. 3, 9, or in presence of large external magnetic fields making the Fermi surface asymmetric, like in ref. 12.

1.3. The Hubbard Model

The Hubbard model Hamiltonian is given by

$$\begin{aligned} H = & -t \sum_{x\in\Lambda} \sum_{\sigma=\pm} (a^+_{x,\sigma} a^-_{x+1,\sigma} + a^+_{x+1,\sigma} a^-_{x,\sigma}) \\ & + U \sum_{x\in\Lambda} a^+_{x,+} a^-_{x,+} a^+_{x,-} a^-_{x,-} - \mu \sum_{x\in\Lambda} \sum_{\sigma=\pm} a^+_{x,\sigma} a^-_{x,\sigma} \end{aligned} \tag{1.1}$$

where Λ is an interval of L points on the one dimensional lattice of step 1, which will be chosen equal to $(-[L/2], [(L-1)]/2)$ and $a^{\pm}_{x,\sigma}$ is a set of fermionic creation or annihilation operators with spin $\sigma = \pm$satisfying periodic boundary conditions; $t = 1/2$ is the hopping parameter, $U > 0$ is the coupling and μ is the chemical potential. The Hamiltonian verifyes an $SU(2)$ spin symmetry.

A generalization of the above hamiltonian including the effect of a short-ranged (instead of a nearest-neighbor) interaction, and the presence of a magnetic field, is the following

$$H = -\frac{1}{2}\sum_{x\in\Lambda,\sigma}(a^+_{x,\sigma}a^-_{x+1,\sigma}+a^+_{x+1,\sigma}a^-_{x,\sigma})+U\sum_{x,y\in\Lambda}v(x-y)a^+_{x,+}a^-_{x,+}a^+_{y,-}a^-_{y,-}$$
$$-\mu\sum_{x\in\Lambda,\sigma}a^+_{x,\sigma}a^-_{x,\sigma}+h\sum_{x\in\Lambda}(a^+_{x,+}a^-_{x,+}-a^+_{x,-}a^-_{x,-}) \qquad (1.2)$$

When the interaction is given by $Uv(x-y)=U\delta_{x,y}+V\delta_{x+1,y}$, the model is known as the *U–V model.*

We consider the operators $a^\pm_{\mathbf{x},\sigma}=e^{Hx_0}a^\pm_{x,\sigma}e^{-Hx_0}$, $\mathbf{x}=(x,x_0)$ and x_0 will be called time variable.

Many physical properties of the fermionic system at inverse temperature β can be expressed in terms of the *Schwinger functions*, that is the truncated expectations in the Grand Canonical Ensemble of the time order product of the field $a^\pm_{\mathbf{x},\sigma}$ at different space-time points. If

$$\langle X\rangle_{L,\beta}=\frac{Tre^{-\beta H}X}{Tre^{-\beta H}} \qquad (1.3)$$

the Schwinger functions are defined as, if $\varepsilon=\pm$

$$S_{L,\beta}(\mathbf{x}_1,\varepsilon_1,\sigma_1;\dots;\mathbf{x}_n,\varepsilon_n,\sigma_n)=\langle a^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1}\dots a^{\varepsilon_1}_{\mathbf{x}_n,\sigma_n}\rangle_{L,\beta} \qquad (1.4)$$

We will denote by $S(\mathbf{x}_1,\dots)$ the $\lim_{L,\beta\to\infty}$ of (1.4). An important role is played by the two point Schwinger function

$$S_{L,\beta}(\mathbf{x},+,\sigma;\mathbf{y},-,\sigma)=S_{L,\beta}(\mathbf{x},\mathbf{y}) \qquad (1.5)$$

Denoting by $\hat S_{L,\beta}(k,x_0)$ the Fourier transform of $S_{L,\beta}(\mathbf{x})$ with respect to the x variable, $n_k\equiv\hat S_{L,\beta}(k,0^-)$ is the *occupation number*, the average number of particles with momentum k. Another important physical quantity is the density-density *correlation function*

$$\Omega_{\mathbf{x},\mathbf{y}}=\langle\rho_\mathbf{x}\rho_\mathbf{y}\rangle-\langle\rho_\mathbf{x}\rangle\langle\rho_\mathbf{y}\rangle \qquad (1.6)$$

where $\rho_\mathbf{x}=\sum_{\sigma=\pm}a^+_{\mathbf{x},\sigma}a^-_{\mathbf{x},\sigma}$.

1.4. The Non Interacting $U=0$ Case

The two point Schwinger function in the non interacting case is given by, in the limit $L,\beta\to\infty$

$$S_0(\mathbf{x},\mathbf{y})=\int_{-\infty}^{\infty}dk_0\int_{-\pi}^{\pi}dk\frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0+\mu-\cos k} \tag{1.7}$$

where $\mathbf{k}=(k,k_0)$. It is easy to check that one can write, if $\mu=\cos p_F^0$ and $v_0=\sin p_F^0$

$$S_0(\mathbf{x},\mathbf{y})=\sum_{\omega=\pm}\frac{e^{i\omega p_F^0(x-y)}}{v_0x_0+i\omega x}+\bar{g}(\mathbf{x},\mathbf{y}) \tag{1.8}$$

with $|\bar{g}(\mathbf{x},\mathbf{y})|\leqslant\frac{C}{1+|\mathbf{x}-\mathbf{y}|^{1+\theta}}$, θ a positive constant; that is, the two point Schwinger function decays as $O(|\mathbf{x}-\mathbf{y}|^{-1})$ oscillating with period $\frac{\pi}{p_F^0}$. Important physical properties are:

(1) The occupation number is given by $n_k=\chi(|\mathbf{k}|\leqslant p_F^0)$, that is it is discontinuous.

(2) The bidimensional Fourier transform of the density correlation function has singularities at $(\pm 2p_F^0,0)$ and $(0,0)$; in $(\pm 2p_F^0,0)$ it has a logarithmic singularity while in $(0,0)$ the Fourier transform is bounded.

(3) The one dimensional Fourier transform at $x_0=0$ of the density correlation is continuous, while its first derivative in k has a first order discontinuity in $k=0,\pm 2p_F^0$.

1.5. Main Result

Our result can be informally stated in the following way

In the not half filled band case and in the weak coupling regime, the (repulsive) Hubbard model (1.1) *is a Luttinger liquid.*

A more formal statement is the following theorem.

1.6.

Theorem 1. Consider the hamiltonian (1.1) with $-1<\mu<1$ and $\mu\neq 0$ (not filled or half filled band case); there exists an $\varepsilon>0$ such that, for $0\leqslant U\leqslant\varepsilon$

(a) the two point Schwinger function (1.5) is given by, in the limit $L,\beta\to\infty$

$$S(\mathbf{x},\mathbf{y})=\sum_{\omega=\pm}\frac{e^{i\omega p_F(x-y)}}{v(x_0-y_0)+i\omega(x-y)}\frac{1+A_\omega(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^\eta}+\bar{S}(\mathbf{x},\mathbf{y}) \tag{1.9}$$

with

$$\eta=aU^2+U^2f_0(U)\quad p_F=\cos^{-1}\mu+f_1(U)\quad v=v_0+f_2(U) \tag{1.10}$$

where $a>0$, $|f_0(U)|,|f_1(U)|,|f_2(U)|\leqslant CU$ and

$$|\bar{\partial}_x^{n_1}\partial_{x_0}^{n_0}A_\omega(\mathbf{x},\mathbf{y})|\leqslant CU\frac{1}{|\mathbf{x}-\mathbf{y}|^{n_0+n_1}}\quad |\bar{S}(\mathbf{x},\mathbf{y})|\leqslant\frac{C}{1+|\mathbf{x}-\mathbf{y}|^{1+\theta}} \tag{1.11}$$

for suitable positive constants C,θ, if $\bar{\partial}$ denotes the discrete derivative. Moreover the occupation number n_k is continuous at $k=\pm p_F$ but its first derivative diverges at $k=\pm p_F$ as $|k-(\pm p_F)|^{-1+\eta}$.

(b) The density-density correlation function (1.6) can be written as

$$\Omega_{\mathbf{x},\mathbf{0}}=\cos(2p_Fx)\Omega^a(\mathbf{x})+\Omega^b(\mathbf{x})+\Omega^c(\mathbf{x}), \tag{1.12}$$

with

$$\begin{aligned}\Omega^a(\mathbf{x})&=\frac{1+A_1(\mathbf{x})}{2\pi^2[x^2+(vx_0)^2]^{1+\eta_1}},\\ \Omega^b(\mathbf{x})&=\frac{1}{2\pi^2[x^2+(vx_0)^2]}\left\{\frac{x_0^2-(x/v_0)^2}{x^2+(v_0x_0)^2}+A_2(\mathbf{x})\right\},\end{aligned} \tag{1.13}$$

$$|A_i(\mathbf{x})|\leqslant CU\qquad |\Omega^c(\mathbf{x})|\leqslant\frac{C}{1+|\mathbf{x}|^{2+\theta}}, \tag{1.14}$$

for some constant C, where $\eta_1=-bU+Uf_4(U)$ with $b>0$ and $|f_4(U)|\leqslant CU$. Finally for $\alpha=1,2$ and if C_{n_0,n_1} is a constant

$$|\bar{\partial}_x^{n_1}\partial_{x_0}^{n_0}A_\alpha(\mathbf{x})|\leqslant\frac{C_{n_0,n_1}}{1+|\mathbf{x}|^{n_0+n_1}} \tag{1.15}$$

(c) Let $\hat{\Omega}(\mathbf{k})$, $\mathbf{k}=(k,k_0)\in[-\pi,\pi]\times\mathbb{R}^1$, the Fourier transform of $\Omega_{\mathbf{x},0}$. Then near $\mathbf{k}=(0,0)$

$$|\hat{\Omega}(\mathbf{k})|\leqslant c_2[1+U\log|\mathbf{k}|^{-1}] \tag{1.16}$$

and, at $\mathbf{k}=(\pm 2p_F,0)$, $\hat{\Omega}(\mathbf{k})$ diverges as

$$|\mathbf{k}-(\pm 2p_F,0)|^{2\eta_1}/|\eta_1| \tag{1.17}$$

Let $G(x)=\Omega_{\mathbf{x},0}|_{x_0=0}$ and $\hat{G}(k)$ its Fourier transform. Then $\hat{G}(k)$ is bounded and $\partial_k\hat{G}(k)$ has a first order discontinuity at $k=0$, with a jump equal to $1+O(U)$, and, at $k=\pm 2p_F$, diverges as $|k-(\pm 2p_F)|^{2\eta_1}$; for $k\neq 0,\pm 2p_F$ $\partial_k G(k)$ is bounded.

Remarks.

(a) A naive estimate of ε in the above Theorem is $\varepsilon=O(|\mu|^{\alpha})$ for some constant α, for μ close to 0; that is U must be taken smaller and smaller as we are closer and closer to the half filled band case.

(b) A first effect of the interaction is that the Fermi momentum p_F is modified by the interaction by $O(U)$ terms.

(c) More dramatic is the effect of the interaction on the long distance asymptotic behavior of the physical observables; it turns out that the two point Schwinger function decays *faster* in presence of the interaction, while the correlation function decays *slower*. The large distance decay is power law with anomalous critical indexes depending non trivially by the coupling U.

(d) As a consequence the occupation number n_k, which in the non interacting case have a discontinuity at $k=\pm p_F$, has no discontinuity in presence of the interaction; this proves that the $d=1$ Hubbard model is a *Luttinger liquid* in the sense of ref. 16. The lack of discontinuity in the occupation number can be physically interpreted saying that fermionic quasiparticles are not present.

The interaction changes the log-singularity at $\mathbf{k}=(\pm 2p_F,0)$ of the bidimensional Fourier transform of the density correlation in a power law singularity, with a nonuniversal critical index $O(U)$. This enhancement of the singularity is considered a signal of the tendency of the system to develop density wave excitations with period π/p_F, generically incommensurate with the lattice. On the other hand the singularity in $\mathbf{k}=(0,0)$ is much weaker, that is at most logarithmic.

In the same way, the interaction leaves invariant the singularity of the first derivative of the one dimensional Fourier transform of the correlations in $k=0$ (a first order discontinuity) while the singularity in $k=\pm 2p_F$ is changed by the interaction from a discontinuity to a power law singularity.

(e) The two points Schwinger function and the density correlation can be written as sum of two terms; one which is very similar to corresponding quantities in the Luttinger model, and in which the dependence from p_F is quite simple (they can be written as oscillating terms times terms which are free of oscillations, in the sense that each derivative increases the decay by a unit, see (1.11), (1.15)) and another (non Luttinger like) in which the dependence on p_F and the lattice steps is very complicate; this last term decays faster than the Luttinger like terms but the derivatives do not increase the decay for the presence of oscillating terms. The non Luttinger like terms have Fourier transform which is bounded; however sufficiently high derivatives of the Fourier transform can be singular for values different from $k=0, \pm p_F, \pm 2p_F$ (such singularities were indeed observed in numerical simulations, see ref. 21).

(f) Our results provide a proof of Luttinger liquid behavior, but they are still not enough accurate to prove an important property called *spin-charge separation*, which is believed true for the Hubbard model; namely that the asymptotic behavior of the two point Schwinger function is $(x_0+iv_c x)^{-\frac{1}{2}-\eta_c}(x_0+iv_s x)^{-\frac{1}{2}-\eta_s}$, with $v_c - v_s = O(U)$ and $\eta_c, \eta_s = O(U)$; (1.9),(1.11) is compatible with such behavior but is not enough accurate to prove it. Another property which could be probably proved by an extension of our techniques is the Borel summability of our critical indexes as a function of U.

(g) Finally, we could consider a short range instead of local potential, that is (1.2) with $h=0$. In such a case the condition $U>0$ is replaced by the condition $U\hat{v}(2p_F)+F(U)>0$, where $F(U)$ is a suitable $O(U^2)$ function. Note that the linear term is vanishing for sufficiently long range interactions such that $\hat{v}(2p_F)=0$.

1.7. The Hubbard Model in a Magnetic Field

Let us consider the Hamiltonian (1.2) with $h \neq 0$; the presence of a magnetic field destroys the spin rotation invariance. Moreover it turns out that one can consider also attractive interactions, if the interaction is smaller than the magnetic field. Calling $S_{\sigma,L,\beta}(\mathbf{x},\mathbf{y}) = \langle \psi^-_{\mathbf{x},\sigma}\psi^+_{\mathbf{y},\sigma}\rangle_{L,\beta}$ we prove the following result.

1.8.

Theorem 2. Consider the hamiltonian (1.2) with $-1<\mu<1$ and $0\leqslant h\leqslant h_0$ for a suitable constant h_0; assume also that $\cos^{-1}(\mu+h)+\cos^{-1}(\mu-h)\neq\pi$. There exists positive constants $\varepsilon_1,\varepsilon_2$ (depending on μ and h, and ε_2 vanishing as $h\to 0$) such that, if $-\varepsilon_2\leqslant U\leqslant\varepsilon_1$ the two point Schwinger function is given by, in the limit $L,\beta\to\infty$

$$S_\sigma(\mathbf{x},\mathbf{y})=\sum_\omega \frac{e^{i\omega p_F^\sigma(x-y)}}{v(x_0-y_0)+i\omega(x-y)}\frac{1+A_\omega(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^\eta}+\bar{S}(\mathbf{x},\mathbf{y}) \tag{1.18}$$

with

$$\begin{gathered}\eta=aU^2+O(U^3)\quad p_F^\sigma=\cos^{-1}(\mu+\text{sign}\,(\sigma)h)+O(U)\\ v=\sin(\cos^{-1}(\mu+\text{sign}\,(\sigma)h))+O(U)\end{gathered} \tag{1.19}$$

where $a>0$ and

$$|\partial^{n_0}\bar{\partial}^{n_1}A(\mathbf{x},\mathbf{y})|\leqslant CU|\mathbf{x}-\mathbf{y}|^{-n_0-n_1}\quad |\bar{S}(\mathbf{x},\mathbf{y})|\leqslant\frac{C}{1+|\mathbf{x}-\mathbf{y}|^{1+\theta}} \tag{1.20}$$

for suitable positive constants C,θ.

The other statements in the previous theorem can be repeated with some obvious modifications. The above result says that the Hubbard model is still a Luttinger liquid even in presence of a magnetic field; this happens even in the attractive case, if the interaction is smaller than the magnetic field.

1.9. Contents

In Sections 2 and 3 we write the Hubbard model (1.1) partition function as a Grassmann integral, and we define a multiscale integration procedure; we get an expansion in terms of running coupling constants, whose regularity properties are stated in Theorem 3. In Section 4 we study the flow of the running coupling constants and in Sections 5 and 6 we derive the cancellations of the Hubbard model Beta function by Ward identities and Correction identities of a suitable reference model. Finally, such results are applied in Section 7 to the computation of the Schwinger functions and the correlations and in Section 8 the presence of the magnetic field is included. We rely on many technical results already obtained in refs. 4–7 (the presence of spin has a small effect on the proof of convergence, for instance) and we focus mainly on the difference with respect to the spinless case.

2. THE ULTRAVIOLET INTEGRATION

We assume $\mu\in\Omega\cap(-1,1)$, where $\Omega^c=\{\mu:|\mu|,|\mu\pm 1|\leqslant\bar\mu\}$, where $\bar\mu>0$ is a fixed constant. We call $\mathcal{D}_{L,\beta}\equiv\mathcal{D}_L\times\mathcal{D}_\beta$, with $\mathcal{D}_L\equiv\{k=2\pi n/L, n\in\mathbb{Z}, -[L/2]\leqslant n\leqslant[(L-1)/2]\}$ and $\mathcal{D}_\beta\equiv\{k_0=2(n+1/2)\pi/\beta, n\in Z, -M\leqslant n\leqslant M-1\}$; moreover we define

$$\tilde t=1-\delta\quad \tilde t\cos p_F=\mu-\nu \tag{2.1}$$

with δ,ν suitable counterterms to be fixed properly in the following. We introduce a finite set of Grassmanian variables $\{\hat\psi^\pm_{\mathbf{k}}\}$, one for each $\mathbf{k}\in\mathcal{D}_{L,\beta}$, and a linear functional $P(d\psi)$ on the generated Grassmannian algebra, such that

$$\begin{aligned}\int P(d\psi)\hat\psi^-_{\mathbf{k}_1,\sigma_1}\hat\psi^+_{\mathbf{k}_2,\sigma_2}&=L\beta\delta_{\mathbf{k}_1,\mathbf{k}_2}\delta_{\sigma_1,\sigma_2}\hat g(\mathbf{k}_1),\\ \hat g(\mathbf{k})&=\frac{1}{-ik_0+\tilde t\cos p_F-\tilde t\cos k}\,.\end{aligned}\tag{2.2}$$

We will call $\hat g(\mathbf{k})$ the *propagator* of the field.

We define also *Grassmanian field* $\psi^\pm_{\mathbf{x}}$ is defined by

$$\psi^\pm_{\mathbf{x},\sigma}=\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\hat\psi^\pm_{\mathbf{k},\sigma}e^{\pm i\mathbf{k}\cdot\mathbf{x}} \tag{2.3}$$

such that

$$\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\hat g(\mathbf{k})=\int P(d\psi)\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}\equiv g^{L,\beta}(\mathbf{x};\mathbf{y})\,, \tag{2.4}$$

It is well known that the partition function $Z=\langle e^{-\beta H}\rangle_{L,\beta}$ can be rewritten as the limit $M\to\infty$ of the Grassmann integral

$$\int P(d\psi)e^{-\mathcal{V}} \tag{2.5}$$

where $P(d\psi)$ is the Grassmann integration with propagator (2.4) and

$$\begin{aligned}\mathcal{V}=U\int_{-\beta/2}^{\beta/2}\mathrm{d}x_0\sum_{x\in\Lambda}\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},-}\\ +\nu\int_{-\beta/2}^{\beta/2}dx_0\sum_{x\in\Lambda,\sigma}\psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}+\delta\int_{-\beta/2}^{\beta/2}dx_0\sum_{x,y\in\Lambda,\sigma}t_{x,y}\psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}\end{aligned}\tag{2.6}$$

where $t_{x,y}=\frac{1}{2}\delta_{y,x+1}+\frac{1}{2}\delta_{x,y+1}$. Let T^1 be the one dimensional torus, $\|k-k'\|_{T^1}$ the usual distance between k and k' in T^1 and $\|k\|=\|k-0\|$. We introduce a *scaling parameter* $\gamma>1$ and a positive function $\chi(\mathbf{k}')\in C^\infty(T^1\times R)$, $\mathbf{k}'=(k',k_0)$, such that

$$\chi(\mathbf{k}')=\chi(-\mathbf{k}')=\begin{cases}1 & \text{if } |\mathbf{k}'|<t_0\equiv a_0 v_0/\gamma,\\ 0 & \text{if } |\mathbf{k}'|>a_0 v_0,\end{cases} \tag{2.7}$$

where

$$|\mathbf{k}'|=\sqrt{k_0^2+(v_0\|k'\|_{T^1})^2}\,, \tag{2.8}$$

$$a_0=\min\{p_F/2,(\pi-p_F)/2\}\,, \tag{2.9}$$

The definition (2.7) is such that the supports of $\chi(k-p_F,k_0)$ and $\chi(k+p_F,k_0)$ are disjoint and the C^∞ function on $T^1\times R$

$$\hat f_{u.v.}(\mathbf{k})\equiv 1-\chi(k-p_F,k_0)-\chi(k+p_F,k_0) \tag{2.10}$$

is equal to 0, if $[v_0\|(|k|-p_F)\|_{T^1}]^2+k_0^2<t_0^2$. We define

$$g^{L,\beta}(\mathbf{x};\mathbf{y})=g^{u.v.}(\mathbf{x},\mathbf{y})+g^{i.r.}(\mathbf{x},\mathbf{y}) \tag{2.11}$$

with

$$g^{u.v.}(\mathbf{x},\mathbf{y})=\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{\hat f_{u.v.}(\mathbf{k})}{-ik_0-\tilde t\cos k+\tilde t\cos p_F}$$

$$g^{i.r.}(\mathbf{x},\mathbf{y})=\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{\prod_{\omega=\pm1}\chi(k-\omega p_F,k_0)}{-ik_0-\tilde t\cos k+\tilde t\cos p_F} \tag{2.12}$$

From the integration over $\psi^{(u.v.)}$ we get

$$e^{-L\beta E_{L,\beta}}=e^{-L\beta\tilde E_1}\int P(d\psi^{(i.r.)})\,e^{-\mathcal{V}^{(0)}(\psi^{(i.r.)})},\quad \mathcal{V}^{(0)}(0)=0\,, \tag{2.13}$$

$$e^{-\mathcal{V}^{(0)}(\psi^{(i.r.)})-L\beta\tilde E_1}=\int P(d\psi^{(u.v.)})e^{-\mathcal{V}(\psi^{(i.r.)}+\psi^{(u.v)})}\,. \tag{2.14}$$

We will call $\psi^{(i.r.)}=\psi^{(\leqslant 0)}$; $\mathcal{V}^{(0)}(\psi^{(\leqslant 0)})$ can be written in the form

$$\mathcal{V}^{(0)}(\psi^{(\leqslant 0)}) = \sum_{n=1}^{\infty} \frac{1}{(L\beta)^{2n}} \sum_{\underline{\sigma}} \sum_{\mathbf{k}_1,\dots,\mathbf{k}_{2n}} \prod_{i=1}^{2n} \hat{\psi}^{(\leqslant 0)\varepsilon_i}_{\mathbf{k}_i,\sigma_i} \times \hat{W}^{(0)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{k}_1,\dots,\mathbf{k}_{2n-1})\, \delta\left(\sum_{i=1}^{2n} \varepsilon_i \mathbf{k}_i\right), \tag{2.15}$$

where $\underline{\sigma}=(\sigma_1,\dots,\sigma_{2n})$, $\underline{\omega}=(\omega_1,\dots,\omega_{2n})$ and we used the notation

$$\delta(\mathbf{k})=\delta(k)\delta(k_0)\,, \quad \delta(k)=L\sum_{n\in\mathbb{Z}}\delta_{k,2\pi n}\,, \quad \delta(k_0)=\beta\delta_{k_0,0}. \tag{2.16}$$

We prove in the Appendix that

$$|\hat{W}^{(0)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{k}_1,\dots,\mathbf{k}_{2n-1})| \leqslant L\beta C \max(U,|\nu|)^{\max(1,n/2)} \tag{2.17}$$

The $SU(2)$ spin invariance implies that the quartic terms have the following form

$$\frac{1}{(L\beta)^4} \sum_{\mathbf{k}_1,..,\mathbf{k}_4} W^{(0)}_{4,\underline{\omega}}(\mathbf{k}_1,\dots,\mathbf{k}_4) \sum_{\sigma} \delta\left(\sum_i \varepsilon_i \mathbf{k}_i\right) \times[\psi^+_{\mathbf{k}_1,\sigma}\psi^-_{\mathbf{k}_2,\sigma}\psi^+_{\mathbf{k}_3,\sigma}\psi^-_{\mathbf{k}_4,\sigma}+\psi^+_{\mathbf{k}_1,\sigma}\psi^-_{\mathbf{k}_2,\sigma}\psi^+_{\mathbf{k}_3,-\sigma}\psi^-_{\mathbf{k}_4,-\sigma}] \tag{2.18}$$

where $W^{(0)}_{4,\underline{\omega}}$ is spin independent.

3. THE INFRARED INTEGRATION

3.1. Quasiparticles

We define also, for any integer $h\leqslant 0$,

$$f_h(\mathbf{k}')=\chi(\gamma^{-h}\mathbf{k}')-\chi(\gamma^{-h+1}\mathbf{k}'); \tag{3.1}$$

we have, for any $\bar{h} < 0$,

$$\chi(\mathbf{k}') = \sum_{h=\bar{h}+1}^{0} f_h(\mathbf{k}') + \chi(\gamma^{-\bar{h}}\mathbf{k}') . \tag{3.2}$$

Note that, if $h \leqslant 0$, $f_h(\mathbf{k}') = 0$ for $|\mathbf{k}'| < t_0\gamma^{h-1}$ or $|\mathbf{k}'| > t_0\gamma^{h+1}$, and $f_h(\mathbf{k}') = 1$, if $|\mathbf{k}'| = t_0\gamma^h$. We finally define, for any $h \leqslant 0$:

$$\hat{f}_h(\mathbf{k}) = f_h(k - p_F, k_0) + f_h(k + p_F, k_0); \tag{3.3}$$

This definition implies that, if $h \leqslant 0$, the support of $\hat{f}_h(\mathbf{k})$ is the union of two disjoint sets, A_h^+ and A_h^-. In A_h^+, k is strictly positive and $\|k - p_F\|_{T^1} \leqslant a_0\gamma^h \leqslant a_0$, while, in A_h^-, k is strictly negative and $\|k + p_F\|_{T^1} \leqslant a_0\gamma^h$. The label h is called the *scale* or *frequency* label. Note that, if $\mathbf{k} \in \mathcal{D}_{L,\beta}$, then $|\mathbf{k} \pm (p_F, 0)| \geqslant \sqrt{(\pi\beta^{-1})^2 + (v_0\pi L^{-1})^2}$, by the definition of $\mathcal{D}_{L,\beta}$. Therefore

$$\hat{f}_h(\mathbf{k}) = 0 \quad \forall h < h_{L,\beta} = \min\{h: t_0\gamma^{h+1} > \sqrt{(\pi\beta^{-1})^2 + (v_0\pi L^{-1})^2}\}, \tag{3.4}$$

and, if $\mathbf{k} \in \mathcal{D}_{L,\beta}$, the definitions (2.10) and (3.3), together with the identity (3.2), imply that

$$1 = \sum_{h=h_{L,\beta}}^{0} \hat{f}_h(\mathbf{k}) + \hat{f}_{u.v.}(\mathbf{k}) . \tag{3.5}$$

We now introduce, for any $h \leqslant 0$, a set of Grassmann variables $\psi^{\pm}_{\mathbf{k}',\omega}$ such that

$$\int P(d\psi^{(h)}) \psi^{-(h)}_{\mathbf{k}'_1,\omega,\sigma} \psi^{+(h)}_{\mathbf{k}'_2,\omega',\sigma'} = L\beta \delta_{\sigma,\sigma'} \delta_{\omega,\omega'} \delta_{\mathbf{k}'_1,\mathbf{k}'_2} g^{(h)}_{\omega}(\mathbf{k}'_1) . \tag{3.6}$$

where

$$g^{(h)}_{\omega}(\mathbf{k}') = \frac{f_h(k', k_0)}{-ik_0 - \tilde{t}\cos(k' + \omega p_F) + \tilde{t}\cos p_F} \tag{3.7}$$

We introduce also the Grassmann variables

$$\psi^{\pm(h)}_{\mathbf{x},\omega,\sigma} = \frac{1}{L\beta} \sum_{\mathbf{k}' \in \mathcal{D}_{L,\beta}} \hat{\psi}^{\pm(h)}_{\mathbf{k}',\omega,\sigma} e^{\pm i\mathbf{k}'\cdot\mathbf{x}} \tag{3.8}$$

so that

$$\int P(d\psi^{(h)})\psi^{-(h)}_{\mathbf{x},\omega,\sigma}\psi^{+(h)}_{\mathbf{y},\omega',\sigma'} = \delta_{\sigma,\sigma'}\delta_{\omega,\omega'}g^{(h)}_{\omega}(\mathbf{x},\mathbf{y}). \tag{3.9}$$

where

$$g^{(h)}_{\omega}(\mathbf{x},\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})}g^{(h)}_{\omega}(\mathbf{k}'). \tag{3.10}$$

It holds that

$$\int P(d\psi^{(i.r.)})\psi^{-(i.r.)}_{\mathbf{x},\sigma}\psi^{+(i.r.)}_{\mathbf{y},\sigma'} = \delta_{\sigma,\sigma'}\sum_{h=h_{L,\beta}}^{0}\sum_{\omega=\pm} e^{-i\omega p_F(x-y)}g^{(h)}_{\omega}(\mathbf{x},\mathbf{y}). \tag{3.11}$$

The above identity implies that, if $F(\psi^{(i.r.)})$ is any analytic function of the variables $\psi^{(i.r)}$

$$\int P(d\psi^{(i.r.)})F(\psi^{(i.r.)}) = \int\prod_{h=h_{L,\beta}}^{0} P(d\psi^{(h)})F\left(\sum_{h=h_{L,\beta}}^{0}\sum_{\omega=\pm} e^{-i\omega p_F x}\psi^{(h)}_{\mathbf{x},\sigma}\right) \tag{3.12}$$

We define also

$$C_h^{-1}(\mathbf{k}) = \sum_{k=-\infty}^{h}\hat{f}_k(\mathbf{k}) \tag{3.13}$$

3.2. Multiscale Integration

The integration of the infrared part is done in an iterative way. Assume that we have integrated the scales $0,-1,\ldots,h+1$ and that we have found

$$\int P_{Z_h,C_h}(d\psi^{(\leqslant h)})e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)})-L\beta E_h}, \quad \mathcal{V}^{(h)}(0)=0, \tag{3.14}$$

where

$$P_{Z_h,C_h}(d\psi^{(\leqslant h)}) = \mathcal{N}_h^{-1} \prod_{\mathbf{k}'\in D} \prod_{\omega=\pm 1} d\psi^{+(\leqslant h)}_{\mathbf{k}',\omega} d\psi^{-(\leqslant h)}_{\mathbf{k}',\omega}$$
$$\exp - \frac{1}{\beta L} \sum_{\substack{\mathbf{k}'\in D \\ C_h^{-1}(\mathbf{k})>0}} Z_h C_h(\mathbf{k}) \psi^{+(\leqslant h)}_{\mathbf{k}',\omega,\sigma} \tag{3.15}$$
$$\times(-ik_0 + \omega v_0 \sin k' + \cos p_F (\cos k' - 1))\psi^{-(\leqslant h)}_{\mathbf{k}',\omega,\sigma}$$

and

$$\mathcal{V}^{(h)}(\psi^{(\leqslant h)}) = \sum_{n=1}^{\infty} \frac{1}{(L\beta)^{2n}} \sum_{\underline{\sigma}} \sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_{2n}} \prod_{i=1}^{2n} \hat\psi^{(\leqslant 0)\varepsilon_i}_{\mathbf{k}'_i,\omega_i,\sigma_i}$$
$$\times \hat W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{k}'_1, \dots, \mathbf{k}'_{2n-1})\, \delta\left(\sum_{i=1}^{2n} \varepsilon_i \mathbf{k}'_i + \sum_{i=1}^{2n} \varepsilon_i \omega_i p_F\right), \tag{3.16}$$

and in particular the quartic terms have the following form

$$\frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,..,\mathbf{k}'_4} \sum_{\underline{\omega}} W^{(h)}_{4,\underline{\omega}}(\mathbf{k}'_1, \dots, \mathbf{k}'_3) \sum_{\sigma} \delta\left(\sum_i \varepsilon_i \mathbf{k}'_i + \sum_i \varepsilon_i \omega_i p_F\right)$$

$$[\psi^{+}_{\mathbf{k}_1,\omega_1,\sigma}\psi^{-}_{\mathbf{k}_2,\omega_2,\sigma}\psi^{+}_{\mathbf{k}_3,\omega_3,\sigma}\psi^{-}_{\mathbf{k}_4,\omega_4,\sigma} + \psi^{+}_{\mathbf{k}_1,\omega_1,\sigma}\psi^{-}_{\mathbf{k}_2,\omega_2,\sigma}\psi^{+}_{\mathbf{k}_3,\omega_3,-\sigma}\psi^{-}_{\mathbf{k}_4,\omega_4,-\sigma}] \tag{3.17}$$

Note that there exists a scale $\bar h$ such that, for $h \leqslant \bar h$ are present in (3.17) only the monomials verifying

$$\sum_{i=1}^{4} \varepsilon_i \omega_i p_F = 0\,. \tag{3.18}$$

In fact by the compact support properties of the propagators $\|\sum_i \varepsilon_i \mathbf{k}'_i\|_{T^1} \leqslant 4a_0 v_0 \gamma^{h+1}$ and if (3.18) is not satisfied $\|\sum_{i=1}^4 \varepsilon_i \omega_i p_F\|_{T^1} \geqslant C|p_F - \frac{\pi}{2}|$ as the condition $|\mu| \geqslant \bar\mu$ surely implies that $|p_F - \frac{\pi}{2}| > 0$, for U small enough (than $O(\bar\mu)$); hence $\bar h = O(\log|p_F - \frac{\pi}{2}|)$.

3.3. The Localization Operator

We split the effective potential $\mathcal{V}^{(h)}$ as $\mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, where $\mathcal{R} = 1 - \mathcal{L}$ and $\mathcal{L}$, the *localization operator*, is a linear operator on functions of the form (3.16), defined in the following way by its action on the kernels $\hat{W}^{(h)}_{2n,\underline{\omega}}$.

(1) If $2n = 4$ we define

$$\mathcal{L}\hat{W}^{(h)}_{4,\underline{\sigma},\underline{\omega}}(\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}'_3) = L^{-1}\delta\left(\sum_{i=1} \varepsilon_i \omega_i p_F\right) \hat{W}^{(h)}_{4,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}), \tag{3.19}$$

where $\bar{\mathbf{k}}_{\eta\eta'} = (\eta\pi L^{-1}, \eta'\pi\beta^{-1})$.

(2) If $2n = 2$ (in this case there is a non zero contribution only if $\omega_1 = \omega_2$)

$$\mathcal{L}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{k}') = \frac{1}{4}\sum_{\eta,\eta'=\pm1} \hat{W}^{(j)}_{2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})\left\{1 + \eta\frac{L}{\pi} + \eta'\frac{\beta}{\pi}k_0\right\}, \tag{3.20}$$

(3) In all the other cases

$$\mathcal{L}\hat{W}^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{k}'_1, \dots, \mathbf{k}'_{2n-1}) = 0. \tag{3.21}$$

In the not half filled band case $p_F \neq \frac{\pi}{2}$ the condition $\delta(\sum_{i=1} \varepsilon_i\omega_i p_F) \neq 0$ is equivalent to the condition $\sum_{i=1}^4 \varepsilon_i\omega_i \neq 0$. Then the action of $\mathcal{L}$ if $n = 2$ is non trivial only if $\sum_{i=1}^4 \varepsilon_i\omega_i = 0$ and there are only the following possibilities for $\omega_1, \omega_2, \omega_3, \omega_4$:

$$(\omega, \omega, -\omega, -\omega); \quad (\omega, -\omega, -\omega, \omega); \quad (\omega, \omega, \omega, \omega) \tag{3.22}$$

In the half filled band case $p_F = \frac{\pi}{2}$ the action of $\mathcal{L}$ is non trivial also if $\omega_1 = \omega_3 = -\omega_2 = -\omega_4$.

We get, in the not half filled band case

$$\begin{aligned}\mathcal{L}\mathcal{V}^{(h)}(\psi) &= \gamma^h n_h F^{(h)}_\nu(\psi) + z_h F^{(h)}_z(\psi) + a_h F^{(h)}_a(\psi) + \gamma_{1,h}F^{(h)}_1(\psi)\\ &\quad + \gamma_{2,h}F^{(h)}_2(\psi) + \gamma_{4,h}F^{(h)}_4(\psi)\end{aligned} \tag{3.23}$$

where

$$F_\nu = \frac{1}{\beta L}\sum_{\mathbf{k}'}\sum_{\omega,\sigma}\psi^+_{\mathbf{k}',\omega,\sigma}\psi^-_{\mathbf{k}',\omega,\sigma}$$
$$F_z = \frac{1}{\beta L}\sum_{\mathbf{k}'}(-ik_0)\sum_{\omega,\sigma}\psi^+_{\mathbf{k}',\omega,\sigma}\psi^-_{\mathbf{k}',\omega,\sigma}$$
$$F_a = \frac{1}{\beta L}\sum_{\mathbf{k}'}[\omega\sin p_F\sin k' + \cos p_F(\cos k'-1)]\sum_{\omega,\sigma}\psi^+_{\mathbf{k}',\omega,\sigma}\psi^-_{\mathbf{k}',\omega,\sigma}$$
$$F_1 = \frac{1}{(\beta L)^4}\sum_{\mathbf{k}'_1,\mathbf{k}'_2,\mathbf{k}'_3,\mathbf{k}'_4}\sum_{\omega}\sum_{\sigma,\sigma'}\delta\left(\sum_i \varepsilon_i\mathbf{k}'_i\right)\psi^+_{\mathbf{k}'_1,\omega,\sigma}\psi^-_{\mathbf{k}'_2,-\omega,\sigma}\psi^+_{\mathbf{k}'_3,-\omega,\sigma'}\psi^-_{\mathbf{k}'_4,\omega,\sigma'}$$
$$F_2 = \frac{1}{(\beta L)^4}\sum_{\mathbf{k}'_1,\mathbf{k}'_2,\mathbf{k}'_3,\mathbf{k}'_4}\sum_{\omega,\sigma,\sigma'}\delta\left(\sum_i \varepsilon_i\mathbf{k}'_i\right)\psi^+_{\mathbf{k}'_1,\omega,\sigma}\psi^-_{\mathbf{k}'_2,\omega,\sigma}\psi^+_{\mathbf{k}'_3,-\omega,\sigma'}\psi^-_{\mathbf{k}'_4,-\omega,\sigma'}$$
$$F_4 = \frac{1}{(\beta L)^4}\sum_{\mathbf{k}'_1,\mathbf{k}'_2,\mathbf{k}'_3,\mathbf{k}'_4}\sum_{\omega,\sigma}\delta\left(\sum_i \varepsilon_i\mathbf{k}'_i\right)\psi^+_{\mathbf{k}'_1,\omega,\sigma}\psi^-_{\mathbf{k}'_2,\omega,\sigma}\psi^+_{\mathbf{k}'_3,\omega,\sigma'}\psi^-_{\mathbf{k}'_4,\omega,\sigma'}$$

Note that

$$\begin{aligned}\gamma_{2,h} &= \hat{W}^{(h)}(\omega p_F, \omega p_F, -\omega p_F, -\omega p_F)\\ \gamma_{1,h} &= \hat{W}^{(h)}(\omega p_F, -\omega p_F, -\omega p_F, \omega p_F)\\ \gamma_{4,h} &= \hat{W}^{(h)}(\omega p_F, \omega p_F, \omega p_F, \omega p_F)\end{aligned} \tag{3.24}$$

and in particular

$$\gamma_{4,0} = U\hat{v}(0) + O(U^2) \quad \gamma_{2,0} = U\hat{v}(0) + O(U^2) \quad \gamma_{1,0} = U\hat{v}(2p_F) + O(U^2)$$

In the case of local interactions $v(p) = 1$. Note also that the spin symmetric part of $\gamma_{4,h}$ is vanishing by Pauli principle.

3.4. Renormalization

We write (3.14) as

$$\int P_{Z_h,C_h}(d\psi^{(\leqslant h)})\, e^{-\mathcal{L}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}) - \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}) - L\beta E_h}, \tag{3.25}$$

and we include the quadratic part of $\mathcal{L}\mathcal{V}^{(h)}$ given by $z_h \int d\mathbf{k}' \sum_{\omega,\sigma} \psi^+_{\mathbf{k}',\omega,\sigma}$ $(-ik_0 + \omega \sin k' + \cos p_F(\cos k' - 1))\psi^-_{\mathbf{k}',\omega,\sigma}$ in the free integration; we call

$$\mathcal{L}\bar{\mathcal{V}}^h = \mathcal{L}\mathcal{V}^{(h)} - z_h \int d\mathbf{k}' \sum_{\omega,\sigma} \psi^+_{\mathbf{k}',\omega,\sigma} \times (-ik_0 + \omega \sin k' + \cos p_F(\cos k' - 1))\psi^-_{\mathbf{k}',\omega,\sigma} \tag{3.26}$$

so that we obtain

$$\int P_{\tilde{Z}_{h-1},C_h}(d\psi^{(\leqslant h)})\, e^{-\mathcal{L}\bar{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(\leqslant h)}) - \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}) - L\beta E_h}, \tag{3.27}$$

where

$$\tilde{Z}_{h-1}(\mathbf{k}) \stackrel{\text{def}}{=} Z_h(1 + z_h C_h^{-1}(\mathbf{k})) \tag{3.28}$$

It is convenient to rescale the fields:

$$\begin{aligned}\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}) \stackrel{\text{def}}{=}\; & g_{1,h} F_1(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}) + g_{2,h} F_2(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}) \\ & + g_{4,h} F_1(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}) + \delta_h F_a(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}) \\ & + \gamma^h \nu_h F_\nu(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}) + \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}),\end{aligned} \tag{3.29}$$

where

$$\nu_h = \frac{Z_h}{Z_{h-1}} n_h \quad \delta_h = \frac{Z_h}{Z_{h-1}}[a_h - z_h] \quad g_{i,h} = \left[\frac{Z_h}{Z_h - 1}\right]^2 \gamma_{i,h} \tag{3.30}$$

Finally the r.h.s. of (3.27) can be rewritten as

$$e^{-L\beta t_h} \int P_{Z_{h-1},C_{h-1}}(d\psi^{(\leqslant h-1)}) \int P_{Z_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)})}, \tag{3.31}$$

where

$$Z_{h-1} = Z_h(1 + z_h) \quad \tilde{f}_h(\mathbf{k}') = f_h(\mathbf{k}')\left[1 + \frac{z_h f_{h+1}(\mathbf{k}')}{1 + z_h f_h(\mathbf{k}')}\right] \tag{3.32}$$

and

$$\int P_{Z_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})\psi^{-(h)}_{\mathbf{x},\omega_1,\sigma_1}\psi^{+(h)}_{\mathbf{y},\omega_2,\sigma_2}=\delta_{\sigma_1,\sigma_2}\delta_{\omega_1,\omega_2}\frac{\tilde{g}^{(h)}_\omega(\mathbf{x},\mathbf{y})}{Z_{h-1}}$$
$$=\frac{1}{Z_{h-1}}\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}\tilde{f}_h(\mathbf{k})\frac{e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})}}{-ik_0+\omega v_0\sin k'+\cos p_F(\cos k'-1)}. \tag{3.33}$$

We then integrate $\psi^{(h)}$

$$\int P_{Z_{h-1},\tilde{f}_h^{-1}}(d\psi^{(\leqslant h)})\,e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)})}=e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)})} \tag{3.34}$$

and the procedure can be iterated.

Note that the quartic terms in $\mathcal{L}\mathcal{V}^h$ can be written in coordinate representation in the following way

$$\sum_{\omega,\sigma}\int d\mathbf{x}g_2^h[\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}\psi^+_{\mathbf{x},-\omega,\sigma}\psi^-_{\mathbf{x},-\omega,\sigma}+\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}\psi^+_{\mathbf{x},-\omega,-\sigma}\psi^-_{\mathbf{x},-\omega,-\sigma}]$$
$$+g_1^h[\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},-\omega,\sigma}\psi^+_{\mathbf{x},-\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}+\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},-\omega,\sigma}\psi^+_{\mathbf{x},-\omega,-\sigma}\psi^-_{\mathbf{x},\omega,-\sigma}]$$
$$+g_4^h[\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x}_2,\omega,\sigma}\psi^+_{\mathbf{x},\omega,-\sigma}\psi^-_{\mathbf{x},\omega,-\sigma}] \tag{3.35}$$

where $\int d\mathbf{x}=\int dx_0\sum_x$. Finally note that the propagator is written as

$$\tilde{g}^{(h)}_\omega(\mathbf{x}-\mathbf{y})=g^{(h)}_{\omega,L}(\mathbf{x}-\mathbf{y})+r^{(h)}_\omega(\mathbf{x}-\mathbf{y}) \tag{3.36}$$

where

$$g^{(h)}_{\omega,L}(\mathbf{x}-\mathbf{y})=\frac{1}{\beta L}\sum_{\mathbf{k}}f_h(\mathbf{k})\frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0+\omega k}$$

and for any positive integer N

$$|r^{(h)}_\omega(\mathbf{x}-\mathbf{y})|\leqslant C_N\frac{\gamma^{2h}}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|)^N}$$

It is easy to verify that $g^{(h)}_{\omega,L}$ verifies the same bound of $r^{(h)}_\omega$ with a γ^h less. We call $v_k=(\nu_k,\delta_k,g_{1,k},g_{2,k},g_{4,k})$, $k\leqslant 0$ and $v_1=(\nu,\delta,U)$; moreover we call $g_k=(g_{1,k},g_{2,k}g_{4,h})$ and $\mu_k=(g_{2,k},g_{4,k})$, $k\leqslant 0$. The above integration procedure generates a power series expansion for $W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})$ in (3.16) in terms of the running coupling constants $\vec{v}_k$, $k=1,0,-1,-2,\ldots,h$, which is indeed convergent if they are small enough. More exactly it holds the following result.

3.5.

The following crucial result holds.

Theorem 3. Assume that $\mu \neq 0, \pm 1$ and $\sup_{k \geqslant h} |\vec{v}_k| \leqslant \varepsilon_h$; assume also that, for some constant c, $\sup_{k\geqslant h} \frac{Z_k}{Z_{k-1}} \leqslant e^{c\varepsilon_h^2}$; then there exists $\bar{\varepsilon}$ such that, for $\varepsilon_h \leqslant \bar{\varepsilon}$ the functions $W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ are analytic in the running coupling constants $(\vec{v}_k)_{k\geqslant h}$ and, for a suitable constant C, α

$$\int d\mathbf{x}_1 \dots d\mathbf{x}_n |W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1, \dots, \mathbf{x}_n)| \leqslant (C\varepsilon_h \bar{h}^\alpha)^{\max(1,n/2)} L\beta\gamma^{(2-n)h} \quad (3.37)$$

Sketch of the proof. The proof is essentially identical to the one of Theorem (3.12) of ref. 4 about the spinless case. The only important difference is that there exists a finite scale $\bar{h} = O(\log|p_F - \frac{\pi}{2}|)$ such that for $h \leqslant \bar{h}$ there are no contributions to the effective potential $\bar{\mathcal{V}}^h$ (3.16) with $n = 2$ and a choice of ω, ε such that (3.18) is not verified. We can write the effective potential $\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)})$, for $h \leqslant 0$, in terms of a *tree expansion*, similar to that described in ref. 4.

We need some definitions and notations.

(1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geqslant 1$ points, the *endpoints* of the *unlabeled tree* (see Fig. 1), so that r is not a branching point.

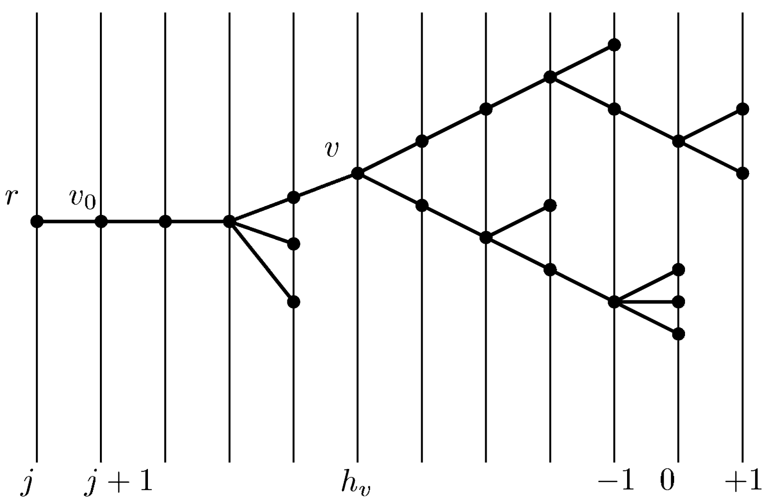

Fig. 1. A tree τ and its labels.

n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order.

Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n.

We shall consider also the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

(2) We associate a label $h \leqslant 0$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an an integer taking values in $[h, 2]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index h. There is the constraint that, if v is an endpoint, $h_v > h+1$.

The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$. Given a vertex v, which is not an endpoint, $\mathbf{x}_v$ will denote the family of all space-time points associated with one of the endpoints following v. Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $h+1$. Finally, if there is only one endpoint, its scale must be equal to $+2$ or $h+2$.

(3) With each endpoint v of scale $h_v = +2$, we associate one of the three contributions to $\mathcal{V}(\psi^{(\leqslant 1)})$, written as in (2.6) and a set $\mathbf{x}_v$ of space-time points, the corresponding integration variables. With each endpoint v of scale $h_v \leqslant 1$ we associate one of local terms in $\mathcal{L}\mathcal{V}^{(h_v-1)}$ (3.29); we will say that the endpoint is of type g_1, g_2 and so on depending on the term we associate to it.

Moreover, we impose the constraint that, if v is an endpoint and $\mathbf{x}_v$ is a single space-time point (that is the corresponding term is local), $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v.

(4) If v is not an endpoint, the *cluster* L_v with frequency h_v is the set of endpoints following the vertex v; if v is an endpoint, it is itself a

(*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.

(5) We introduce a *field label* f to distinguish the field variables appearing in the terms associated with the endpoints as in item (3); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\sigma(f)$ and $\omega(f)$ will denote the space-time point, the σ index and the ω index, respectively, of the field variable with label f.

If $h \leqslant 0$, the effective potential can be written in the following way, see ref. 4:

$$\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}) + L\beta\tilde{E}_{h+1} = \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{h,n}} V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leqslant h)}), \tag{3.38}$$

where, if v_0 is the first vertex of τ and $\tau_1, \dots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0,

$V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leqslant h)})$ is defined inductively by the relation

$$\begin{aligned} V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leqslant h)}) = {} & \frac{(-1)^{s+1}}{s!}\mathcal{E}^T_{h+1}[\bar{V}^{(h+1)} \\ & \times(\tau_1, \sqrt{Z_h}\psi^{(\leqslant h+1)}); ..; \bar{V}^{(h+1)}(\tau_s, \sqrt{Z_h}\psi^{(\leqslant h+1)})], \end{aligned} \tag{3.39}$$

and $\bar{V}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\leqslant h+1)})$

(a) is equal to $\mathcal{R}\hat{\mathcal{V}}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\leqslant h+1)})$ if the subtree τ_i is not trivial;

(b) if τ_i is trivial and $h \leqslant -1$, it is equal to one of the terms in $\mathcal{L}\mathcal{V}^{(h+1)}$ (3.29) or, if $h = 0$, to one of the terms contributing to $\hat{\mathcal{V}}(\psi^{<1})$ (2.6).

It is then easy to get, by iteration of the previous procedure, a simple expression for $V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\leqslant h)})$, for any $\tau \in \mathcal{T}_{h,n}$.

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \dots, v_{s_v}$ are the vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i}\backslash Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$.

Given $\tau\in\mathcal{T}_{h,n}$, there are many possible choices of the subsets P_v, $v\in\tau$, compatible with all the constraints; we shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. Then we can write

$$V^{(h)}(\tau,\sqrt{Z_h}\psi^{(\leqslant h)})=\sum_{\mathbf{P}\in\mathcal{P}_\tau} V^{(h)}(\tau,\mathbf{P})\,; \tag{3.40}$$

Calling $W^{(h)}_{\tau,\mathbf{P}}$ the kernels of $V^{(h)}(\tau,\mathbf{P})$ (see (3.16)) and repeating the analysis in Section 3 of ref. 4 one gets the following bound (analogous to (3.105) of ref. 4)

$$\begin{aligned}\int d\mathbf{x}_{v_0}|W^{(h)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})|\leqslant{}& C^nL\beta\varepsilon_h^n\gamma^{-hD_k(P_{v_0})}\\ &\times\prod_{v\text{ not e.p.}}\chi(P_v)\left\{\frac{1}{s_v!}C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}\left(Z_{h_v}/Z_{h_v-1}\right)^{|P_v|/2}\gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]}\right\},\end{aligned} \tag{3.41}$$

where $z(P_v)=2$ if $|P_v|=2$ and $z(P_v)=1$ if $|P_v|=1$ and $\left\|\sum_{f\in P_v}\varepsilon(f)\omega(f)p_F\right\|_{T^1}=0$; moreover $\chi(P_v)$ are defined so that $\chi(P_v)=0$ if $|P_v|=4$, $h_v\leqslant\bar{h}$ and $\left\|\sum_{f\in P_v}\varepsilon(f)\omega(f)p_F\right\|_{T^1}\neq 0$, and $\chi(P_v)=1$ otherwise.

We call $2-\frac{|P_v|}{2}-z(P_v)$ the *dimension* of the vertex v in the tree. If no renormalization is defined $\mathcal{R}=1$ then one gets a similar bound with $z_v(P_v)=0$. Hence if $\mathcal{R}=1$ the vertices v with $|P_v|=4$ have vanishing dimension (*marginal terms*) while if $|P_v|=2$ they have positive dimension (*relevant terms*). The presence of the χ-functions in (3.41) is easily understood by noting that one can insert freely such χ functions in momentum space, then one passes to coordinate space and make bounds using the Gram-Hadamard inequality as in ref. 4.

For any v such that $h_v\leqslant\bar{h}$ it holds $-2+\frac{|P_v|}{2}+z(P_v)\geqslant 1$, that is the dimension is negative, while if $h_v\geqslant\bar{h}$ it holds $-2+\frac{|P_v|}{2}+z(P_v)\geqslant 0$.

We have to perform the sums over τ and $\mathbf{P}$. The number of unlabeled trees is $\leqslant 4^n$; fixed an unlabeled tree, the number of terms in the sum over the various labels of the tree is bounded by C^n, except the sums over the scale labels and the sets $\mathbf{P}$.

In order to bound the sums over the scale labels and **P** we first use the inequality, for a constant $0<c<1$

$$\prod_{v \text{ not e.p.}} \left(Z_{h_v}/Z_{h_v-1}\right)^{|P_v|/2} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]}$$

$$\leqslant \left[\prod_{\tilde{v}} (\chi(h_{\tilde{v}} \leqslant \bar{h})\gamma^{-c(h_{\tilde{v}}-h_{\tilde{v}'})} + \chi(h_{\tilde{v}} \geqslant \bar{h}))\right] \left[\prod_{v \text{ not e.p.}} \chi(|P_v|>4)\gamma^{-\frac{|P_v|}{40}}\right], \tag{3.42}$$

where $\tilde{v}$ are the non trivial vertices, and $\tilde{v}'$ is the non trivial vertex immediately preceding $\tilde{v}$ or the root. Then it holds that, noting the number of nontrivial vertices is bounded by n

$$\sum_{\{h_{\tilde{v}}\}} \left[\prod_{\tilde{v}} \left(\chi(h_{\tilde{v}} \leqslant \bar{h})\gamma^{-c(h_{\tilde{v}}-h_{\tilde{v}'})} + \chi(h_{\tilde{v}} \geqslant \bar{h})\right)\right] \leqslant C^n |\bar{h}|^{\alpha n} \tag{3.43}$$

for some numerical constant α. Finally the sum over **P** can be done as described in ref. 4. ▮

Remark. By (3.42) we get also that the bound for a tree $\tau \in \mathcal{T}_{h,n}$ with at least a vertex at scale k improves by a factor $\gamma^{\theta(h-k)}$; this property is called *short memory property*.

4. THE FLOW EQUATION

4.1. Second Order Analysis

By the iterative integration procedure seen in the previous section it follows that the running coupling constants verify a recursive relation whose r.h.s. is called *Beta function*:

$$\frac{Z_{h-1}}{Z_h} = 1 + z_h(v_h, \ldots, v_1) \tag{4.1}$$

$$\nu_{h-1} = \gamma \nu_h + \beta_\nu^{(h)}(v_h, \ldots, v_1) \tag{4.2}$$

$$\delta_{h-1} = \delta_h + \beta_\delta^{(h)}(v_h, \ldots, v_1) \tag{4.3}$$

$$g_{i,h-1} = g_{i,h} + \beta_{g,i}^{(h)}(v_h, \ldots, v_1) \tag{4.4}$$

with $i=(1,2,3)$. The above equations are also called flow equations. The functions z_h, $\beta_\nu^{(h)}$, $\beta_\delta^{(h)}\beta_{g,i}^{(h)}$ are expressed by the tree expansion seen in Section 3 (for details, see ref. 4). The contribution to $\beta_{g,1}^{(h)}$ from the trees with two end-points associated to the quartic running coupling constants is given by, if $\int d\mathbf{r}=\int_{-\beta/2}^{\beta/2} dr_0 \sum_{r\in\Lambda}$

$$\sum_{k\leqslant h} 4\int d\mathbf{r} g_\omega^{(k)}(\mathbf{r}) g_{-\omega}^{(h)}(-\mathbf{r}) g_{1,k} g_{1,k} = 4\int d\mathbf{r} \sum_{k=h,h+1} g_\omega^{(k)}(\mathbf{r}) g_{-\omega}^{(h)}(-\mathbf{r}) g_{1,h} g_{1,k} \tag{4.5}$$

Using that $g_{1,h}-g_{1,h+1}=O(v_h^2)$ and (3.36), and computing the equation analogue to (4.5) for $g_{2,h-1}$ and $g_{4,h-1}$ we get that $g_{i,h}$ verify the following equations

$$\begin{aligned} g_{1,h-1} &= g_{1,h} - a g_{1,h}^2 + O(\bar{v}_h^2 \gamma^{\theta h}) + O(\bar{v}_h^3) \\ g_{2,h-1} &= g_{2,h} - \tfrac{a}{2} g_{1,h}^2 + O(\bar{v}_h^2 \gamma^{\theta h}) + O(\bar{v}_h^3) \\ g_{4,h-1} &= g_{4,h} + O(\bar{v}_h^2 \gamma^{\theta h}) + O(\bar{v}_h^3) \end{aligned} \tag{4.6}$$

with a a positive constant, given by

$$a=a_1+a_2=4\int d\mathbf{r}[g_{L,\omega}^{(h)}(\mathbf{r}) g_{L,-\omega}^{(h)}(\mathbf{r}) + g_{L,\omega}^{(h)}(\mathbf{r}) g_{L,-\omega}^{(h+1)}(\mathbf{r})] \tag{4.7}$$

If we neglect the cubic contributions $O(\bar{v}_h^3)$ it is easy to see that the flow is bounded (in sense that the quartic running coupling remain smaller than $O(U)$ for any h) if $U>0$; in the general case in which the interaction is non local the conditions is $g_{1,0}=Uv(2p_F)+O(U^2)>0$. By taking into account all higher order terms could destroy such behavior; aim of the following sections is to prove that also taking into account the full Beta function the quartic running coupling remain smaller than $O(U)$.

4.2. Beta Function Decomposition

We have two free parameters at our disposal, ν and δ; we will show that we can fix them so that $\nu_h=O(U^2\gamma^{\tau h})$ and $\delta_h=O(U^2\gamma^{\tau h})$. We fix then our attention on the flow equation for $g_{1,h}, g_{2,h}, g_{4,h}$.

More explicitly (4.4) can be written as

$$
\begin{aligned}
g_{1,h-1} &= g_{1,h} - g_{1,h}[a_1 g_{1,h} + a_2 g_{1,h+1}] \\
&\quad + G_h^1(g_h,\ldots,g_0) + \sum_{k,k'} g_{1,k} g_{1,k'} H^1_{h,k,k'}(v_h,\ldots,v_0) + R_h^1(v_h,\ldots,v_1) \\
g_{2,h-1} &= g_{2,h} - \frac{1}{2} g_{1,h}[a_1 g_{1,h} + a_2 g_{1,h+1}] + \beta_h^2(\mu_h,\ldots,\mu_0) + G_h^2(g_h,\ldots,g_0) \\
&\quad + \sum_{k,k'} g_{1,k} g_{1,k'} H^2_{h,k,k'}(v_h,\ldots,v_0) + R_h^2(v_h,\ldots,v_1) \qquad (4.8) \\
g_{4,h-1} &= g_{2,h} + \beta_h^4(\mu_h,\ldots,\mu_0) + G_h^4(g_h,\ldots,g_0) \\
&\quad + \sum_{k,k'} g_{1,k} g_{1,k'} H^4_{h,k,k'}(v_h,\ldots,v_0) + R_h^4(v_h,\ldots,v_1)
\end{aligned}
$$

where the following definitions are used:

(1) We write in (4.1) $z_k = z_k^1 + z_k^2$, where z_k^1 is defined iteratively as the sum of all trees with only end-points at scale $\leqslant 0$ and with propagators $g^{(k)}_{L,\omega}$, see (3.36), and in which $\frac{Z_{k'-1}}{Z_{k'}}$, $k' \geqslant k$ is replaced by $1 + z_{k'}^1$.

(2) The functions $\beta_h^2, \beta_h^4, G_h^2, G_h^4, G_h^1, g_1 g_1 H^i$, with $i = 1, 2, 4$ are the sum of all the trees with only end-points at scale $\leqslant 0$ and with propagators $g_L^{(k)}$, see (3.36), and in which the factors $\frac{Z_{k-1}}{Z_k}$, $k \geqslant h$ are replaced by $1 + z_k^1$.

(3) The terms contributing to β_h^2, β_h^4 are by definition independent from $g_{1,k}, k \geqslant h$.

(4) The terms contributing to G_h^1, G_h^2, G_h^4 by definitions depend linearly from $g_{1,k}$, that is they are vanishing if $g_{1,k} = 0$ for any k and their second derivatives respect to $g_{1,k}$ are also vanishing, while the first derivative are not vanishing.

(5) The terms at least quadratic in g_1 are included in $\sum_{k,k'} g_{1,k} g_{1,k'} H^i_{h,k,k'}$ and by the short memory property

$$
|H^i_{h,k,k'}| \leqslant C \bar{v}_h \gamma^{\theta(h-k)} \gamma^{\theta(h-k')} \qquad (4.9)
$$

(6) In $R_h^{(i)}$ we include; terms depending from ν_h or δ_h; terms with at least a propagator $r_1^h(\mathbf{x}-\mathbf{y})$, see (3.36); or terms with at least an endpoint at scale 1.

Note that the above decomposition is obtained by an analogous decomposition over trees, so that the determinant bounds of Section 3 are still valid.

In writing (4.8) we have used that the beta function contributing to g_1 has at least a g_1; in fact consider a contribution to the antiparallel part of g_1; it is not invariant under the transformation $\psi^{\pm}_{1,\sigma} \to e^{\pm\sigma}\psi^{\pm}_{1,\sigma}$ and $\psi^{\pm}_{-1,\sigma} \to \psi^{\pm}_{-1,\sigma}$ while the terms corresponding to g_2 and g_4 are invariant.

The flow given by (4.8) is very difficult to study; luckily dramatic cancellations appear, given by, if $\bar{g}_h = \max_{k\geqslant h}(|g^1_k| + |g^2_k| + |g^4_k|)$ and $\bar{\mu}_h = \max_{k\geqslant h}(|g^2_k| + |g^4_k|)$, the following result.

Theorem 4. (Partial vanishing of the Beta function).
The functions $\beta^2_h, \beta^4_h, G^2_h, G^4_h, G^1_h$, for $|\nu_h| \leqslant \varepsilon$ are such that, for a suitable constants C, θ

$$|\beta^2_h(\mu_h, \dots, \mu_h)\;| \leqslant C\bar{\mu}^2_h\gamma^{\theta h} \quad |\beta^4_h(\mu_h, \dots, \mu_h)| \leqslant C\bar{\mu}^2_h\gamma^{\theta h} \tag{4.10}$$

$$|G^2_h(g_h, \dots, g_h)\;| \leqslant C\bar{g}^2_h\gamma^{\theta h} \quad |G^4_h(g_h, \dots, g_h)| \leqslant C\bar{g}^2_h\gamma^{\theta h} \tag{4.11}$$

$$|\;G^1_h(g_h, \dots, g_h)| \leqslant C\bar{g}^2_h\gamma^{\theta h} \tag{4.12}$$

The above lemma says that a dramatic cancellation happens in the series for the above functions; each order is sum of many terms $O(1)$, but at the end the final sum is $O(\gamma^{\theta h})$, that is asymptotically vanishing. We call such property *partial vanishing of the Beta function* (partial because the $O(g^2_1)$ terms are not vanishing).

By the above lemma, which will proved in the following two sections as consequence of suitable *Ward identities*, we can prove that the flow is bounded for any $g_{1,0} > 0$. Note that in ref. 8 a proof of (1.10) using the exact solution of the Mattis model was sketched; (4.11) and (4.12) were assumed without proof.

We proceed in the following way. We first *assign* a sequence $\underline{\nu} \stackrel{\text{def}}{=} \{\nu_h\}_{h\leqslant 1}$, $\underline{\delta} \stackrel{\text{def}}{=} \{\delta_h\}_{h\leqslant 1}$ not necessarily solving the flow equation for ν, δ, but such that $|\nu_h|, |\delta_h| \leqslant cU\gamma^{\theta h}$, for any $h \leqslant 1$. We then solve the flow equation for $g_{i,h}$, parametrically in ν, δ, and show that, *for any sequence* $\underline{\nu}$, $\underline{\delta}$ with the supposed property, the solution $\underline{g}(\underline{\nu}, \underline{\delta}) = \{g_{1,h}(\underline{\nu}, \underline{\delta}), g_{2,h}(\underline{\nu}, \underline{\delta}), g_{4,h}(\underline{\nu}, \underline{\delta})\}_{h\leqslant 1}$ exists and has good decaying properties. We finally fix the sequence $\underline{\nu}$ via a convergent iterative procedure.

Lemma 1. Assume that $|\nu_h|, |\delta_h| \leqslant cU\gamma^{\theta h}$ for any h. For $U > 0$ and small enough the flow is given by, for any h

$$|g_{2,h} - g_{2,0} - g_{1,0}/2| \leqslant U^{3/2} \quad |g_{4,h} - g_{4,0}| \leqslant U^{3/2} \quad 0 < g_{1,h} \leqslant \frac{g_{1,0}}{1 - a/3 g_{1,0} h} \tag{4.13}$$

Proof. By using that $|\nu_h|, |\delta_h| \leqslant cU\gamma^{\theta h}$ it holds that

$$|R^i_h| \leqslant CU^2\gamma^{\theta h} \tag{4.14}$$

It is convenient to introduce $\tilde{g}_{2,h} = 2g_{2,h} - g_{1,h}$; then using (4.10) and (4.14)

$$\tilde{g}_{2,h-1} = \tilde{g}_{2,h} + \sum_{k \geqslant h} D_{h,k} + \sum_{k \geqslant h} (2D^2_{h,k} - D^1_{h,k}) + \sum_{k,k'} g_{1,k} g_{1,k'} \bar{H}_{h,k,k'} + \bar{R}_h \tag{4.15}$$

with

$$D_{h,k} = \beta^2_h(\mu_h, \dots, \mu_h, \mu_k, \mu_{k+1}, \dots, \mu_0) - \beta^2_h(\mu_h, \dots, \mu_h, \mu_h, \mu_{k+1}, \dots, \mu_0) \tag{4.16}$$
$$D^i_{h,k} = G^i_h(g_h, \dots, g_h, g'_k, g_{k'+1}, \dots, g_0) - G^i_h(g_h, \dots, g_h, g_h, g_{k'+1}, \dots, g_0) \quad i = 1, 2$$

and a similar equation for $g_{4,h}$; $\bar{H}_{h,k,k'}$ verifies (4.9), $\bar{R}_h$ (4.14) and

$$|D_{h,k}| \leqslant C\gamma^{-2\theta(k-h)} U |g_h - g_k| \qquad |D^i_{h,k}| \leqslant CU\gamma^{-2\theta(k-h)} |g_h - g_k| \tag{4.17}$$

Assume that for $k > h$

$$0 \leqslant g_{1,k-1} \leqslant \frac{g_{1,0}}{1 - a/3 g_{1,0}(k-1)} \quad |g_{k-1} - g_k| \leqslant \left[U^{\frac{5}{4}} \gamma^{\theta k} + \left[\frac{g_{1,0}}{1 - a/3 g_{1,0} k} \right]^2 \right] \tag{4.18}$$

We have then to prove that such inequalities hold for $k = h - 1$. Noting that

$$\sum_{k=h}^{-1} \gamma^{\theta(h-k)} \frac{1}{-k} = \frac{1}{-h} \sum_{k=h}^{-1} \gamma^{\theta(h-k)} + \sum_{k=h}^{-1} \gamma^{\theta(h-k)} \frac{(k-h)}{kh} \leqslant \frac{C_1}{-h} \tag{4.19}$$

we obtain

$$\sum_{k,k'} g_{1,k} g_{1,k'} \bar{H}^2_{h,k,k'} \leqslant C U g^2_{1,h} \tag{4.20}$$

Moreover

$$\begin{aligned}\sum_{k\geqslant h} |D_{h,k}| &\leqslant \sum_{k=-1}^{h} C C_1 U \gamma^{-2\theta(k-h)} \sum_{k'=h}^{k} \left| U^{\frac{5}{4}} \gamma^{\theta k'} + \left[\frac{g_{1,0}}{1-a/3 g_{1,0} k'} \right]^2 \right| \\ &\leqslant C_2 C U \left(U^{\frac{5}{4}} \gamma^{\theta h} + \sum_{k=-1}^{h} |k-h| \gamma^{2\theta(h-k)} \left[\frac{g_{1,0}}{1-a/3 g_{1,0} k} \right]^2 \right)\end{aligned} \tag{4.21}$$

and the last addend can be bounded by

$$\sum_{k=-1}^{h} \gamma^{\theta(h-k)} \frac{1}{k^2} \leqslant C_2 \left[\frac{1}{1-a/3 g_{1,0} h} \right]^2 . \tag{4.22}$$

Then by (4.15) we get

$$|\tilde{g}_{2,h-1} - \tilde{g}_{2,h}| \leqslant C_3 \left(U^2 \gamma^{\theta h} + U \left(\frac{g_{1,0}}{1-\frac{a}{3} g_{1,0} h} \right)^2 \right) \tag{4.23}$$

and

$$|\tilde{g}_{2,h-1} - \tilde{g}_{2,0}| \leqslant C_3 \sum_{k=h}^{0} \left(U^2 \gamma^{\theta k} + U \left[\frac{g_{1,0}}{1-a/3 g_{1,0} k} \right]^2 \right) \leqslant U^{3/2} \tag{4.24}$$

In the same way in the flow for g_4 we use that there are no second order contributions quadratic in $g_{1,h}$. Finally, we write, using (4.8) and the short memory property (namely that $\gamma^{\theta(h-k)} g_{1,k} \leqslant C g_{1,h}$)

$$g_{1,h-1} - g_{1,h} \leqslant -\frac{a}{3} g_{1,h} g_{1,h-1} \tag{4.25}$$

or

$$g_{1,h-1} \leqslant \frac{g_{1,h}}{1+\frac{a}{3}g_{1,h}} \tag{4.26}$$

and as $\frac{x}{1+x}$ is an increasing function and by induction $0 < g_{1,h} \leqslant \frac{g_{1,0}}{1-\frac{a}{3}g_{1,0}h}$ so that

$$g_{1,h-1} \leqslant \frac{g_{1,0}(1-\frac{a}{3}hg_{1,0})^{-1}}{1+\frac{a}{3}g_{1,0}(1-\frac{a}{3}hg_{1,0})^{-1}} \leqslant \frac{g_{1,0}}{1-\frac{a}{3}g_{1,0}(h-1)}. \tag{4.27}$$

Moreover $g_{1,h-1} = g_{1,h}(1+O(U))$ by (4.8), and $g_{1,h} > 0$ so that $g_{1,h-1} > 0$. ∎

4.3. The Choice of the Counterterms

In the previous section we have solved the flow equation for $g_{i,h}$ parametrically in any sequence $\underline{\nu} = \{\nu_h\}_{h\leqslant 1}$, $\underline{\delta} = \{\delta_h\}_{h\leqslant 1}$ such that $|\nu_h| \leqslant cU\gamma^{\theta h}$, $|\delta_h| \leqslant cU\gamma^{\theta h}$ for any h. We show now that indeed we can choose ν, δ so that $\underline{\nu} = \{\nu_h\}_{h\leqslant 1}$, $\underline{\delta} = \{\delta_h\}_{h\leqslant 1}$ verify such a property.

Lemma 2. There exist sequences $\underline{\nu} = \{\nu_h\}_{h\leqslant 1}$, $\underline{\delta} = \{\delta_h\}_{h\leqslant 1}$ such that $|\nu_h| \leqslant cU\gamma^{\theta h}$, $|\delta_h| \leqslant cU\gamma^{\theta h}$.

Proof. It holds that

$$\beta_\delta^{(h)} = \beta_{\delta,a}^{(h)} + \beta_{\delta,b}^{(h)} \tag{4.28}$$

where $\beta_{\delta,a}^{(h)}$ us given by a sum of trees with no end-points ν_k, δ_k and only propagators $g_{L,\omega}^{(k)}$ (3.36); by the symmetry in the exchange x, x_0 of $g_{L,\omega}^{(k)}$, and remembering that $\beta_\delta^{(h)} = \sum_\tau [z(\tau) - a(\tau)]$ it holds that

$$|\beta_{\delta,a}^{(h)}| \leqslant CU\gamma^{2\theta h} \tag{4.29}$$

A similar decomposition can be done also for

$$\beta_\nu^{(h)} = \beta_{\nu,a}^{(h)} + \beta_{\nu,b}^{(h)} \tag{4.30}$$

again with

$$|\beta_{\nu,a}^{(h)}| \leqslant CU\gamma^{2\theta h} \tag{4.31}$$

by the parity property $g^{(h)}_{L,\omega}(\mathbf{x},\mathbf{y})=-g^{(h)}_{L,\omega}(\mathbf{y},\mathbf{x})$. If we want to fix ν,δ in such a way that $\nu_{-\infty}=\delta_{-\infty}=0$, we must have, if $(\nu_1,\delta_1)=(\nu,\delta)$:

$$\nu=-\sum_{k=-\infty}^{1}\gamma^{k-2}\beta^{(k)}_{\nu}(g_k,\delta_k,\nu_k;\dots;g_1,\delta_1,\nu_1). \tag{4.32}$$

$$\delta=-\sum_{k=-\infty}^{1}\beta^{(k)}_{\delta}(g_k,\delta_k,\nu_k;\dots;g_1,\delta_1,\nu_1). \tag{4.33}$$

Note that in (4.32), (4.33) $g_k\equiv g_k(\underline{\nu},\underline{\delta})$.

If we manage to fix ν,δ as in (4.32), (4.33) we also get:

$$\nu_h=-\sum_{k\leqslant h}\gamma^{k-h-1}\beta^{(k)}_{\nu}(g_k,\delta_k,\nu_k;\dots;g_1,\delta_1,\nu_1). \tag{4.34}$$

$$\delta_h=-\sum_{k\leqslant h}\beta^{(k)}_{\delta}(g_k,\delta_k,\nu_k;\dots;g_1,\delta_1,\nu_1). \tag{4.35}$$

Let $\mathfrak{M}_\theta$ be the space of sequences $\underline{\nu}=\{\nu_{-\infty},\dots,\nu_1\}$, $\underline{\delta}=\{\delta_{-\infty},\dots,\delta_1\}$ with small $\|\cdot\|_\theta$ norm, namely the space of sequences $\underline{\nu},\delta$ satisfying:

$$|\delta_k|\leqslant\gamma^{\theta k},\qquad |\nu_k|\leqslant\gamma^{\theta k}$$

We look for a fixed point of the operator $\mathbf{T}:\mathfrak{M}_\theta\to\mathfrak{M}_\theta$ defined as:

$$T(\nu_h)=-\sum_{k\leqslant h}\gamma^{k-h-1}\beta^{(k)}_{\nu}(g_k(\underline{\delta},\underline{\nu}),\nu_k;\dots;g_1,\nu_1). \tag{4.36}$$

$$T(\delta_h)=-\sum_{k\leqslant h}\beta^{(k)}_{\delta}(g_k(\underline{\delta},\underline{\nu}),\delta_k,\nu_k;\dots;g_1,\delta_1,\nu_1). \tag{4.37}$$

First note that, if U is sufficiently small, then $\mathbf{T}$ leaves $\mathfrak{M}_\theta$ invariant: in fact

$$|(\mathbf{T}\nu)_h|\leqslant\sum_{k\leqslant h}2c_1U\gamma^{\theta k}\gamma^{k-h}\leqslant cU\gamma^{\theta h}\quad |(\mathbf{T}\delta)_h|\leqslant\sum_{k\leqslant h}2c_1U\gamma^{\theta k}\leqslant cU\gamma^{\theta h} \tag{4.38}$$

Furthermore we find that $\mathbf{T}$ is a contraction on $\mathfrak{M}_\theta$: in fact

$$|(\mathbf{T}\delta)_h-(\mathbf{T}\delta')_h|\leqslant \sum_{k\leqslant h}|\beta_\delta^{(k)}(g_k(\underline{\nu},\delta),\nu_k,\delta_k;\dots)-\beta_\nu^{(k)}(g_k(\underline{\nu}',\underline{\delta}'),\nu_k',\underline{\delta}_k';\dots)|$$
$$\leqslant c''U\gamma^{\theta h}[\|\underline{\nu}-\underline{\nu}'\|_\theta+\|\underline{\delta}-\underline{\delta}'\|_\theta]. \tag{4.39}$$

and a similar equation holds for ν. Then, a unique fixed point $\underline{\nu}^*,\underline{\delta}$ for $\mathbf{T}$ exists on $\mathfrak{M}_\theta$. ∎

By the above Lemma we have found $\delta(\tilde{t},p_F,U),\nu(\tilde{t},p_F,U)$; inserting them in (2.1) and using the implicit function theorem we get $p_F(U,\mu),\tilde{t}(U,\mu)$.

Finally from an explicit second order computation we obtain that

$$z_h=a[g_{1,h}^2+g_{2,h}^2+g_{4,h}^2]+\beta^{\geqslant 3} \tag{4.40}$$

with $a>0$ is a suitable constant, and using the previous results on the flow of $g_{i,h},\nu_h,\delta_h$ we get $\lim_{h\to-\infty}\frac{Z_h}{\gamma^{\eta h}}=1$, where $\eta=a[g_{2,-\infty}^2+g_{4,-\infty}^2]+O(U^3)$.

5. THE REFERENCE MODEL AND PROOF OF THEOREM 4

5.1. The Model

In order to prove the partial vanishing of the Hubbard model Beta function expressed by (4.10), (4.11), (4.12) we introduce a *reference model* written directly in terms of Grassmann variables, with an ultraviolet cutoff and an infrared cutoff γ^h *with linear dispersion relation and in the continuum*. We study the reference model by Renormalization Group and we show that the Beta function of this model is asymptotically vanishing as a consequence of *Ward identities* due to the formal local chiral gauge invariance (which is however broken by the presence of cutoffs); then we prove that the Beta function of the reference model coincides partly with the Beta function of the Hubbard model, so that we can deduce the *partial vanishing* of the Hubbard model Beta function from the vanishing of the reference model Beta function.

The partition function of the reference model is

$$\int P_L(d\psi)e^{\mathcal{V}_L} \tag{5.1}$$

where the propagator is

$$g^h_{\omega,L}(\mathbf{x}-\mathbf{y}) = \frac{1}{\beta L}\sum_{\mathbf{k}\in\mathcal{D}_{L\beta}} C^{-1}_{h,0}(\mathbf{k})\frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0+\omega k} \tag{5.2}$$

with $C^{-1}_{h,0}(\mathbf{k}) = \sum_{k=-\infty}^{0} f^k(k_0,k)$ and

$$\begin{aligned}\mathcal{V}_L = \sum_\omega \int_{-\beta/2}^{\beta/2} dx_0 \int_{-L/2}^{L/2} dx\, \big[& g_2^o \psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}\psi^+_{\mathbf{x},-\omega,\sigma}\psi^-_{\mathbf{x},-\omega,\sigma} \\ & + g_2^p \psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}\psi^+_{\mathbf{x},-\omega,-\sigma}\psi^-_{\mathbf{x},-\omega,-\sigma} \\ & + g_4 \psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}\psi^+_{\mathbf{x},\omega,-\sigma}\psi^-_{\mathbf{x},\omega,-\sigma}\big]\end{aligned} \tag{5.3}$$

Note that the model is not $SU(2)$ invariant, as the interaction depends from the spin if $g_2^o \neq g_2^p$.

The Grassmann integration can be done by a multiscale analysis essentially identical to the one described in Section 3; however the symmetries of the interaction imply that the local part of the effective potential (3.29) is replaced by

$$\begin{aligned}\mathcal{L}\mathcal{V}^j_L = \sum_\omega \int_{-\beta/2}^{\beta/2} dx_0 \int_{-L/2}^{L/2} dx\, & \tilde g^p_{2,j}\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}\psi^+_{\mathbf{x},-\omega,\sigma}\psi^-_{\mathbf{x},-\omega,\sigma} \\ & + \tilde g^o_{2,j}\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}\psi^+_{\mathbf{x},-\omega,-\sigma}\psi^-_{\mathbf{x},-\omega,-\sigma} \\ & + \tilde g_{4,j}\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x}_2,\omega,\sigma}\psi^+_{\mathbf{x},\omega,-\sigma}\psi^-_{\mathbf{x},\omega,-\sigma}\end{aligned} \tag{5.4}$$

Note in fact that the analogue of ν_h, δ_h are vanishing by (in the limit $L,\beta\to\infty$) parity and invariance in the exchange $(x,x_0)\to(x_0,x)$; moreover, the reference model is invariant under the total gauge transformation $\psi^\pm_{\mathbf{x},\omega,\sigma}\to e^{\pm\alpha_{\omega,\sigma}}\psi^\pm_{\mathbf{x},\omega,\sigma}$ for any values of $\alpha_{\omega,\sigma}$, so that terms of the form $\psi^+_{\omega,\sigma}\psi^-_{,-\omega,\sigma}\psi^+_{-\omega,-\sigma}\psi^-_{\omega,-\sigma}$ cannot be generated in the integration procedure as they violate such symmetry. Note also that, due to the compact support of the cutoff in (5.2), the running coupling constants at scale $k>h$ of the theory with infrared cutoff γ^h or 0 are identical.

It is easy to verify that a tree expansion similar to the one described in Section 3.5 holds also for the reference model, and that the analogue of Theorem 3 holds also in this case. We will prove in Section 6 the following Lemma.

Lemma 3. Assume that $\bar{g} = \max(|g_2^o|, |g_2^p|, |g_4|)$ is small enough; then for any integer $j \leqslant 0$, in the limit $h \to -\infty$ for a suitable constant C

$$|\tilde{g}_{2,j}^o - g_2^o| \leqslant C\bar{g}^2 \quad |\tilde{g}_{2,j}^p - g_2^p| \leqslant C\bar{g}^2 \quad |\tilde{g}_{4,j} - g_p^4| \leqslant C\bar{g}^2 \tag{5.5}$$

Moreover $\tilde{g}_{2,j}^o, \tilde{g}_{2,j}^p, \tilde{g}_{4,j}$ have a limit as $h \to -\infty$.

It is an immediate corollary of Lemma 3 and Theorem 3 that $\vec{v}_k^L = (\tilde{g}_{2,k}^o, \tilde{g}_{2,k}^p, \tilde{g}_{4,k})$ are analytic functions of $\vec{v}_1 = (g_2^p, g_2^p, g_4)$ around $(0, 0, 0)$. Note that analyticity in the coupling around the origin holds for the reference model and *not* for the Hubbard model. Finally the analogue of the flow equations (4.8) is given by, if $\vec{v}_k^L = (\tilde{g}_{2,k}^o, \tilde{g}_{2,k}^p, \tilde{g}_{4,k})$

$$\begin{aligned} \tilde{g}_{2,j-1}^o &= \tilde{g}_{2,j}^o + \tilde{\beta}_j^{2,o}(\vec{v}_j^L, \dots, \vec{v}_0^L) \\ \tilde{g}_{2,j-1}^p &= \tilde{g}_{2,j}^p + \tilde{\beta}_j^{2,p}(\vec{v}_j^L, \dots, \vec{v}_0^L) \\ \tilde{g}_{4,j-1} &= \tilde{g}_{4,j} + \tilde{\beta}_j^4(\vec{v}_j^L, \dots, \vec{v}_0^L) \end{aligned} \tag{5.6}$$

We can rewrite the above equations as, for $j > h$

$$\begin{aligned} \tilde{g}_{2,j-1}^o &= g_{2,j}^o + \tilde{\beta}_j^{2,o}(\vec{v}_j^L, \dots, \vec{v}_j^L) + \sum_{k>j} D_{j,k}^{2,o} \\ \tilde{g}_{2,j-1}^p &= \tilde{g}_{2,j}^p + \tilde{\beta}_j^{2,p}(\vec{v}_j^L, \dots, \vec{v}_j^L) + \sum_{k>j} \tilde{D}_{j,k}^{2,o} \\ \tilde{g}_{4,j-1} &= \tilde{g}_{4,j} + \tilde{\beta}_j^4(\vec{v}_j^L, \dots, \vec{v}_j^L) + \sum_{k>j} D_{j,k}^{2,o} \end{aligned} \tag{5.7}$$

with, for $\alpha = (2, o), (2, p), 4$

$$\tilde{D}_{j,k}^\alpha = \tilde{\beta}_j^\alpha(\vec{v}_j^L, \dots, \vec{v}_j^L, \vec{v}_k^L, \vec{v}_{k+1}^L, \dots, \vec{v}_0^L) - \tilde{\beta}_j^\alpha(\vec{v}_j^L, \dots, \vec{v}_j^L, \vec{v}_j^L, \vec{v}_{j+1}^L, \dots, \vec{v}_0^L) \tag{5.8}$$

5.2. Vanishing of the Reference Model Beta Function

The Beta function is an analytic function of $\vec{v}_j^L$ and it can be written as, if $\alpha = (o, 2), (p, 2), 4$

$$\tilde{\beta}_j^\alpha(\vec{v}_j^L, \dots, \vec{v}_j^L) = \sum_{n_1,n_2,n_3} b_{j,n_1,n_2,n_3}^\alpha (\tilde{g}_{2,j}^o)^{n_1} (\tilde{g}_{2,j}^p)^{n_2} (\tilde{g}_{4,j})^{n_3}$$

We define $n \equiv n_1 + n_2 + n_3$ and $\vec{n} = (n_1, n_2, n_3)$. Note that

$$b^{\alpha}_{j,n_1,n_2,n_3} = b^{\alpha}_{n_1,n_2,n_3} + O(\gamma^{\theta j})$$

Consider $b^{\alpha}_{j,n_1,n_2,n_3}$ and $b^{\alpha}_{k,n_1,n_2,n_3}$ with $k < j$; for any tree τ contributing to β_k there is a tree contributing to β_j; in fact we can perform a change of variables in the propagator $g^i(\mathbf{k})$ respectively $\mathbf{k} \to \gamma^j \bar{\mathbf{k}}$ and $\mathbf{k} \to \gamma^k \bar{\mathbf{k}}$, so that in one case the propagator is $f(\gamma^{-i+k}\bar{\mathbf{k}}) D^{-1}_{\omega}(\bar{\mathbf{k}})$ and in the other $f(\gamma^{-i+j}\bar{\mathbf{k}}) D^{-1}_{\omega}(\bar{\mathbf{k}}) = f(\gamma^{-i+k}\gamma^{-k+j}\bar{\mathbf{k}}) D^{-1}_{\omega}(\bar{\mathbf{k}})$; hence for each tree contributing to b_j there is a tree contributing to b_k, in which the scale of each vertex is shifted by $j-k$; there are extra contributions to b^k with at least a vertex with scale $> k-j$; such trees have the root at scale k so that, by the short memory property, $b_j - b_k = O(\gamma^{\theta j})$, with $0 < \theta < 1$ a suitable constant, and taking the limit $k \to -\infty$ we get $b_j = b + O(\gamma^{\theta j})$.

We will prove the following result.

Lemma 4. Assume that Lemma 3 holds; then for any (n_1, n_2, n_3)

$$b^{\alpha}_{n_1,n_2,n_3} = 0 \tag{5.9}$$

Proof. The proof is by contradiction; assume that, for some $\vec{\bar{n}} = (\bar{n}_1, \bar{n}_2, \bar{n}_3)$ with $\bar{n} = \bar{n}_1 + \bar{n}_2 + \bar{n}_3$

$$\vec{\beta}_j(\vec{v}^L_h, \ldots, \vec{v}^L_h) = \vec{b}_{j,\vec{n}} (\tilde{g}^o_{2,j})^{\bar{n}_1} (\tilde{g}^p_{2,j})^{\bar{n}_2} (\tilde{g}_{4,j})^{\bar{n}_3} + O(\vec{v}^L_j)^{\bar{n}+1}, \tag{5.10}$$

with $\vec{b}_{\vec{\bar{n}}}$ a non vanishing vector, and that for all $n_1 + n_2 + n_3 = n \leqslant \bar{n} - 1$, $\vec{b}_{\vec{n}}$ is vanishing. From Theorem 3 and Lemma 3 $\vec{v}^L_j$ are analytic functions of $\vec{v}_1 = (g^o_2, g^p_2, g_4)$, that is

$$\vec{v}^L_j = \vec{v}_1 + \sum_{n \leqslant \bar{n}} \vec{c}^{(j)}_{\vec{n}} (g^o_2)^{n_1} (g^p_2)^{n_2} (g_4)^{n_3} + O((\vec{v}^L_1)^{\bar{n}+1}) \tag{5.11}$$

and for any fixed j the sequence $\vec{c}^{\,j}_{\vec{n}}$ is a bounded sequence. Inserting (5.11) in the Beta function, using analyticity and equating the coefficient of $(g^o_2)^{n_1} (g^p_2)^{n_2} (g_4)^{n_3}$ with $n_1 + n_2 + n_3 = n \leqslant \bar{n} - 1$ we get

$$\vec{c}^{(j-1)}_{\vec{n}} = \vec{c}^{(j)}_{\vec{n}} + \sum_{k=j+1}^{0} \vec{d}^{\vec{n}}_{j,k} + O(\gamma^{\theta j}) \tag{5.12}$$

where the last sum represents the contribution of $\vec{D}_{j,k}$, so that

$$|\vec{d}^{\vec{n}}_{j,k}| \leqslant \gamma^{-\theta(k-j)} D_{\bar{n}} \sup_{2\leqslant m\leqslant n-1} |\vec{c}^{(j)}_{\vec{m}} - \vec{c}^{(k)}_{\vec{m}}| \tag{5.13}$$

where we have used that $\vec{D}_{j,k}$ is at least quadratic in the running coupling constants, and $D_{\bar{n}}$ is a suitable constant (in j). Note that

$$|\vec{c}^{(j-1)}_{\vec{n}} - \vec{c}_{\vec{n}}| \leqslant \bar{D}_{\bar{n}} \sum_{j'=-\infty}^{j} \left[\left(\sum_{k=j'+1}^{0} \gamma^{-\theta(k-j')} \sup_{2\leqslant m\leqslant n-1} \left| \vec{c}^{(j')}_{\vec{m}} - \vec{c}^{(k)}_{\vec{m}} \right| \right) + \gamma^{\theta j'} \right] \tag{5.14}$$

The above inequality implies by induction that, for $n \leqslant \bar{n}-1$

$$\sup_{2\leqslant m\leqslant n} |\vec{c}^{(k)}_{\vec{m}} - \vec{c}_{\vec{m}}| \leqslant C^n \gamma^{\frac{\theta}{2}k} \tag{5.15}$$

for a suitable C; assume in fact that it is true for $k \geqslant j$ and from (5.14) we get

$$\begin{aligned} |\vec{c}^{(j-1)}_{\vec{n}} - \vec{c}_{\vec{n}}| &\leqslant C^{n-1} \bar{D}_{\bar{n}} \sum_{j'=-\infty}^{j} \left[\sum_{k=j'+1}^{0} \gamma^{-\theta(k-j')} (\gamma^{\frac{\theta}{2}k} + \gamma^{\frac{\theta}{2}j'}) + \gamma^{\theta j'} \right] \\ &\leqslant K C^{n-1} \bar{D}_{\bar{n}} \gamma^{\frac{\theta}{2}(j-1)} \end{aligned} \tag{5.16}$$

so that (5.15) holds for $j-1$ if $C \geqslant K \bar{D}_{\bar{n}}$. On the other hand (5.15) implies

$$|\vec{d}^{\vec{n}}_{j,k}| \leqslant \bar{C}^n \gamma^{\frac{\theta}{2}(j-1)} \tag{5.17}$$

Writing now the analogous of (5.12) for $n = \bar{n}$ we get

$$\vec{c}^{(j-1)}_{\vec{\bar{n}}} = \vec{c}^{(j)}_{\vec{\bar{n}}} + \vec{b}_{j,\vec{n}} + \vec{d}^{\vec{\bar{n}}}_{j,k} \tag{5.18}$$

which can be rewritten as

$$\vec{c}^{(j-1)}_{\vec{\bar{n}}} = \vec{c}^{(j)}_{\vec{\bar{n}}} + \vec{b}_{\vec{\bar{n}}} + O(\gamma^{\frac{\theta}{2}j}) \tag{5.19}$$

so that $\vec{c}^{(j)}_{\vec{\bar{n}}}$ is necessarily a diverging as $j \to \infty$, and this is a contradiction. ∎

5.3. Partial Vanishing of the Hubbard Model Beta Function (Proof of Theorem 4)

We compare the Beta functions of the reference model with the function appearing in the flow equations of the Hubbard model.

(A) Let us start considering first the reference model in the spin symmetric case, that is if $g^o_{2,0}=g^p_{2,0}$. In such a case by the same arguments used in Appendix, for any $k g^o_{2,k}=g^p_{2,k}$ and $\beta^o_{2,k}=\beta^p_{2,k}$, so that the flow equation (5.6) reduces to

$$\tilde g_{2,h-1}=\tilde g_{2,h-1}+\tilde\beta^2_h(\tilde g_{2,h},\tilde g_{4,h};\dots,\tilde g_{2,0},\tilde g_{4,0})$$

$$\tilde g_{4,h-1}=\tilde g_{4,h-1}+\tilde\beta^4_h(\tilde g_{2,h},\tilde g_{4,h};\dots;\tilde g_{2,0},\tilde g_{4,0})$$

It holds that the functions $\tilde\beta^2_h$ and $\tilde\beta^4_h$ essentially coincide with the functions β^2_h,β^4_h of the Hubbard model defined in (4.8); that is, if $\mu_h=(g_{2,h},g_{4,h})$, for a suitable constant C

$$|\tilde\beta^2_h(\mu_h,\dots,\mu_h)-\beta^2_h(\mu_h,\dots\mu_h)|\leqslant C\mu_h^2\gamma^{\theta h} \tag{5.20}$$

$$|\tilde\beta^4_h(\mu_h,\dots,\mu_h)-\beta^4_h(\mu_h,\dots\mu_h)|\leqslant C\mu_h^2\gamma^{\theta h} \tag{5.21}$$

The above equations prove (4.10). In order to prove (5.20) and (5.21) we note that by definition the only difference between $\tilde\beta^2_h,\tilde\beta^4_h$ and β^2_h,β^4_h is that in one case the model is defined on the continuum and in the other case on the lattice. In momentum representation this means that the delta functions in $\tilde\beta$ are defined as $L\beta\delta_{k,0}\delta_{k_0,0}$ while in β are defined as in (2.16). The difference of the two delta functions slightly affects the non local terms on any scale, hence it affects the beta function; however, it is easy to show that this is a negligible phenomenon. Let us consider in fact a particular tree τ and a vertex $v\in\tau$ of scale h_v with $2n$ external fields of space momenta k'_r, $r=1,\dots,2n$; the conservation of momentum implies that $\sum_{r=1}^{2n}\sigma_r k'_r=2\pi m$, with $m=0$ in the continuous model, but m arbitrary integer for the lattice model. On the other hand, k'_r is of order γ^{h_v} for any r, hence m can be different from 0 only if n is of order γ^{-h_v}. Since the number of endpoints following a vertex with $2n$ external fields is greater or equal to $n-1$ and there is a small factor (of order μ_h) associated with each endpoint, we get an improvement, in the bound of the terms with $|m|>0$, with respect to the others, of a factor $\exp(-C\gamma^{-h_v})$.

Hence it is easy to show that the difference between the two beta functions is of order $\mu_h^2 \gamma^{\theta h}$.

(B) In order to prove (4.11) we consider the reference model with $g^o_{2,0} \neq g^p_{2,0}$, so that there are three independent running coupling constants. We have seen that, for $\alpha = (2, p), (2, o), 4$

$$\tilde{\beta}^\alpha_h(v^L_h, \dots, v^L_h) = \sum_{n_1,n_2,n_3} b^\alpha_{h,n_1,n_2,n_3} [g^o_{2,h}]^{n_1} [g^p_{2,h}]^{n_2} [g_{4,h}]^{n_3} \tag{5.22}$$

On the other hand we can write the functions G^α_h (4.8) in the Hubbard model, $\alpha = (2o), (2p), 4$, as

$$G^\alpha_h = \sum_{m_2,m_3} c^\alpha_{h,1,m_2,m_3} [g_{1,h}][g_{2,h}]^{m_2} [g_{4,h}]^{m_3} \tag{5.23}$$

The coefficients $c^\alpha_{h,1,m_2,m_3}$ are given by sum of trees (or product of trees, for the presence of the z^1_k terms) with (in total) one end-point g_1, m_2 end-points g_2 and m_3 end-points g_4; the $SU(2)$ invariance of the Hubbard model implies that $G^{2o}_h = G^{2p}_h$. To g_1 and g_2 correspond two terms, the parallel or antiparallel part, see (3.35), and we can associate to the end-points of the trees contributing to $c^\alpha_{h,1,m_2,m_3}$ an extra index distinguishing the parallel or antiparallel part; then we can write

$$c^\alpha_{h,1,m_2,m_3} = \sum_{m^o_1+m^p_1=1} \sum_{m^o_2+m^p_2=m_2} c^\alpha_{h,m^o_1,m^p_1,m^o_2,m^p_2,m_3} \tag{5.24}$$

It holds that

$$c^\alpha_{h,1,m_2,m_3} = \sum_{m^o_2+m^p_2=m_2} c^\alpha_{h,0,1,m^o_2,m^p_2,m_3} \tag{5.25}$$

that is only the spin parallel part of g_1 can contribute to G^2_h or G^4_h; in fact making the global gauge transformation $\psi^\pm_{1,\sigma} \to e^{i\sigma} \psi^\pm_{1,\sigma}$ and $\psi^\pm_{-1,\sigma} \to \psi^\pm_{-1,\sigma}$, the antiparallel part is not invariant, while the spin parallel (and the g_2, g_4 interactions) are invariant.

Finally note that the spin parallel g_1 interaction is equal (up to a sign) to the spin parallel g_2 interaction, so that, for $\alpha = (2o), (2p), 4$

$$c^\alpha_{0,1,m^o_2,m^p_2,m_3} = -b^\alpha_{m^o_2,m^p_2+1,m_3} = 0 \tag{5.26}$$

(C) It remains to consider (4.12); we can consider equivalently the contribution to the spin parallel or the spin antiparallel, as they are equal by $SU(2)$ invariance of the Hubbard model, that is $G_h^{1o} = G_h^{1p}$. We consider the spin parallel part and we can write

$$G_h^{1p} = \sum_{m_2,m_3} c^{1p}_{h,1,m_2,m_3}[g_{1,h}][g_{2,h}]^{m_2}[g_{4,h}]^{m_3} \tag{5.27}$$

with

$$c^{1p}_{1,m_2,m_3} = \sum_{m_1^o+m_1^p=1} \sum_{m_2^o+m_2^p=m_2} c^{1p}_{m_1^o,m_1^p,m_2^o,m_2^p,m_3}$$

The single g_1 interaction cannot be antiparallel, again because making the global gauge transformation $\psi^{\pm}_{1,\sigma} \to e^{i\sigma}\psi^{\pm}_{1,\sigma}$ and $\psi^{\pm}_{-1,\sigma} \to \psi^{\pm}_{-1,\sigma}$, the antiparallel part is not invariant, while the spin parallel (and the g_2, g_4 interactions) are invariant. Hence

$$c^{1p}_{1,m_2,m_3} = \sum_{m_2^o+m_2^p=m_2} c^{1p}_{0,1,m_2^o,m_2^p,m_3} \tag{5.28}$$

and

$$c^{1p}_{0,1,m_2^o,m_2^p,m_3} = b^{2p}_{m_2^o,m_2^p+1,m_3} = 0 \tag{5.29}$$

as the contribution $(1p)$ and $(2p)$ are identical.

6. WARD IDENTITIES FOR THE REFERENCE MODEL: PROOF OF LEMMA 3

6.1. Dyson Equations

Let us now prove (5.5), extending the analysis in refs. 5–7, to the spinning case. We derive a number of Dyson equations relating some Schwinger functions of the reference model. Let us start from, if $\rho_{\mathbf{p},\omega,\sigma} = \frac{1}{\beta L}\sum_{\mathbf{k}} \psi^{+}_{\mathbf{k},\omega,\sigma}\psi^{-}_{\mathbf{k}-\mathbf{p},\omega,\sigma}$

$$
\begin{aligned}
&\langle \psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\psi^+_{\mathbf{k}_3,-,-\sigma}\psi^-_{\mathbf{k}_4,-,-\sigma}\rangle_T\\
&= g_-(\mathbf{k}_4)\Big\{G^2_-(\mathbf{k}_3)\Big[g^o_2\langle\psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\rho_{\mathbf{k}_1-\mathbf{k}_2,+,\sigma}\rangle_T\\
&\quad +g^p_2\langle\psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\rho_{\mathbf{k}_1-\mathbf{k}_2,+,-\sigma}\rangle_T\\
&\quad + g_4\langle\psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\rho_{\mathbf{k}_1-\mathbf{k}_2,-,-\sigma}\rangle_T\Big]\\
&\quad +\int d\mathbf{p}\Big[g^o_2\langle\psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\psi^+_{\mathbf{k}_3,-,\sigma}\psi^-_{\mathbf{k}_4-\mathbf{p},-,-\sigma}\rho_{\mathbf{p},+,\sigma}\rangle_T\\
&\quad +\int d\mathbf{p} g^p_2\langle\psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\psi^+_{\mathbf{k}_3,-,\sigma}\psi^-_{\mathbf{k}_4-\mathbf{p},-,-\sigma}\rho_{\mathbf{p},+,-\sigma}\rangle_T\\
&\quad +\int d\mathbf{p} g_4\langle\psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\psi^+_{\mathbf{k}_3,+,\sigma}\psi^-_{\mathbf{k}_4-\mathbf{p},+,-\sigma}\rho_{\mathbf{p},-,-\sigma}\rangle_T\Big]\Big\} \qquad (6.1)
\end{aligned}
$$

where

$$G^2_\omega(\mathbf{k}) = \langle\psi^-_{\mathbf{k},\omega,\sigma}\psi^+_{\mathbf{k},\omega,\sigma}\rangle_T \qquad (6.2)$$

Similar Dyson equations holds for $\langle\psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\psi^-_{\mathbf{k}_3,-,\sigma}\psi^-_{\mathbf{k}_4,-,\sigma}\rangle_T$ and $\langle\psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\psi^-_{\mathbf{k}_3,+,\sigma}\psi^-_{\mathbf{k}_4,+,\sigma}\rangle_T$. The Renormalization Group analysis of the preceding sections easily implies (for details, see ref. 6) that, if $|\bar{\mathbf{k}}_i| = \gamma^h, i = 1, 2, 3, 4$

$$
\begin{aligned}
&\langle\psi^+_{\bar{\mathbf{k}}_1,+,\sigma}\psi^-_{\bar{\mathbf{k}}_2,+,\sigma}\psi^+_{\bar{\mathbf{k}}_3,-,-\sigma}\psi^-_{\bar{\mathbf{k}}_4,-,-\sigma}\rangle_T\\
&\quad \equiv G^4_{+,\sigma}(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4) = \gamma^{-4h} Z_h^{-2}[g^o_{2,h} + O(\bar{g}_h^2)] \qquad (6.3)
\end{aligned}
$$

if $\bar{g}_h = \sup_{k\geqslant h}(|g^o_{2,k}| + |g^p_{2,k}| + |g_{4,k}|)$. In the Dyson equations appear the functions

$$G^{2,1}_{\omega,\sigma,\omega',\sigma'}(\mathbf{k}-\mathbf{q},\mathbf{k},\mathbf{q}) = \langle\psi^+_{\mathbf{k},\omega,\sigma}\psi^-_{\mathbf{q},\omega,\sigma}\rho_{\mathbf{k}-\mathbf{q},\omega',\sigma'}\rangle_T \qquad (6.4)$$

$$
\begin{aligned}
&G^{4,1}_{+,\sigma,\omega'\sigma'}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p};\mathbf{p})\\
&\quad = \langle\psi^+_{\mathbf{k}_1,+,\sigma}\psi^-_{\mathbf{k}_2,+,\sigma}\psi^+_{\mathbf{k}_3,-,-\sigma}\psi^-_{\mathbf{k}_4-\mathbf{p},-,-\sigma}\rho_{\mathbf{p},\omega',\sigma'}\rangle_T \qquad (6.5)
\end{aligned}
$$

Either such functions or the Schwinger functions can be obtained by deriving the *Generating functional*

$$\mathcal{W}(\phi, J) = \log \int P_L(d\psi) e^{-V_L(\psi)+\sum_{\omega,\sigma}\int d\mathbf{x}(J_{\mathbf{x},\omega,\sigma}\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}+\phi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}+\psi^+_{\mathbf{x},\omega,\sigma}\phi^-_{\mathbf{x},\omega,\sigma})} \tag{6.6}$$

with respect to the *external fields* $J_{\mathbf{x},\omega,\sigma}$ or $\phi^-_{\mathbf{x},\omega,\sigma}$.

The functions $G^{2,1}, G^{4,1}$ are related by remarkable *Ward Identities* to the Schwinger functions G^2, G^4. In fact, by operating in (6.6) the (local) Gauge transformation $\psi^\pm_{+,\sigma} \to e^{\pm\alpha_\mathbf{x}}\psi^\pm_{+,\sigma}$, $\psi^\pm_{+,-\sigma} \to \psi^\pm_{+,-\sigma}$ and $\psi^\pm_{-,\pm\sigma} \to \psi^\pm_{-,\pm\sigma}$ and deriving with respect to $\phi^+_{\mathbf{y},+,\sigma}, \phi^+_{\mathbf{z},+,\sigma}$, we get, passing to momentum space, the following Ward Identity

$$D_+(\mathbf{p})G^{2,1}_{+,\sigma,+,\sigma}(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2) = G^2_+(\mathbf{k}_1) - G^2_+(\mathbf{k}_2) + \Delta^{2,1}_{+,\sigma,+,\sigma}(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2) \tag{6.7}$$

where

$$\Delta^{2,1}_{\omega,\sigma,\omega',\sigma'}(\mathbf{k}-\mathbf{q},\mathbf{k},\mathbf{q}) = \left\langle \psi^+_{\mathbf{k},\omega,\sigma}\psi^-_{\mathbf{q},\omega,\sigma}\delta\rho_{\mathbf{k}-\mathbf{q},\omega',\sigma'}\right\rangle_T \tag{6.8}$$

$$\delta\rho_{\mathbf{k}-\mathbf{q},\omega,\sigma} = \int d\mathbf{k}' C(\mathbf{k}',\mathbf{k}'-\mathbf{q})\psi^+_{\mathbf{k}',\omega,\sigma}\psi^-_{\mathbf{k}'-\mathbf{k}+\mathbf{q},\omega,\sigma} \tag{6.9}$$

$$C(\mathbf{k}^+,\mathbf{k}^-) = (C_{h,0}(\mathbf{k}^-)-1)D_\omega(\mathbf{k}^-) - (C_{h,0}(\mathbf{k}^+)-1)D_\omega(\mathbf{k}^+) \tag{6.10}$$

The cutoff function $C_{h,0}$ in $P_L(d\psi)$ destroys the local Gauge invariance of the theory, and it is responsible of the *correction term* $\Delta^{2,1}$ in (4.3). As explained in Section 4 of ref. 5. $\bar\Delta^{(ij)} \equiv C(\mathbf{k},\mathbf{q})g^{(i)}(\mathbf{k})g^{(j)}(\mathbf{q})$ is non vanishing if at least one among i or j is 0 or h; this means that either at least one field in $\delta\rho$ is contracted at scale 0, or at least one field in $\delta\rho$ is contracted at scale h. We can split the correction term in the following way

$$\Delta^{2,1}_{\omega,\sigma,\omega',\sigma'} = \Delta^{2,1,\alpha}_{\omega,\sigma,\omega',\sigma'} + \Delta^{2,1,\beta}_{\omega,\sigma,\omega',\sigma'} \tag{6.11}$$

where in $\Delta^{2,1,\alpha}_{\omega,\sigma,\omega',\sigma'}$ there are all the contributions with one of the fields in $\delta\rho$ contracted at scale 0, and $\Delta^{2,1,\beta}_{\omega,\sigma,\omega',\sigma'}$ is the rest. It is easy to check that

$$\left|\Delta^{2,1,\beta}_{\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2)\right|\leqslant \bar{g}_h\frac{\gamma^{-2h}}{Z_h} \tag{6.12}$$

This follows from the bound $|\bar{\Delta}^{(hj)}|\leqslant\gamma^{h-j}\frac{\gamma^{-h-j}}{Z_j}$ and noting that the factor γ^{h-j} gives the correct power counting for the marginal terms linear in J, see ref. 5; note also that the contributions of order 0 in the v^L_h cancels out.

The analysis of $\Delta^{2,1,\alpha}_{\omega,\sigma,\omega',\sigma'}$ is more complex; there are other remarkable identities (first discovered in ref. 6 for the spinless case) called *correction identities* to the functions $G^{2,1}$. It holds in fact the following Lemma.

Lemma 5. There exists functions $\nu_{\omega,\pm\sigma}$ such that $|\nu_{\omega,\pm\sigma}|\leqslant C\bar{g}_h$ and

$$\begin{aligned}\Delta^{2,1,\alpha}_{+,\sigma,+,\sigma} &= \nu^a_{+,\sigma}D_+(\mathbf{p})G^{2,1}_{+,\sigma,+,\sigma}\\ &\quad+\nu^a_{+,-\sigma}D_+(\mathbf{p})G^{2,1}_{+,\sigma,+,-\sigma}+\nu^a_{-,\sigma}D_-(\mathbf{p})G^{2,1}_{+,\sigma,-,\sigma}\\ &\quad+\nu^a_{-,-\sigma}D_-(\mathbf{p})G^{2,1}_{+,\sigma,-,-\sigma}+H^{2,1,\alpha}_{+,\sigma,+,\sigma}\end{aligned} \tag{6.13}$$

with, if $|\bar{\mathbf{k}}_1|=|\bar{\mathbf{k}}_2|=\gamma^h$

$$|H^{2,1,\alpha}_{+,\sigma,+,\sigma}(\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2)|\leqslant C\frac{\gamma^{-2h}}{Z_h^2}\gamma^{\theta h} \tag{6.14}$$

for some constants C and $0<\theta<1$.

The above identity says that the correction $\Delta^{2,1,\alpha}$ can be written in terms of the functions $G^{2,1}$, up to a term which is smaller than $O(\gamma^{\theta h})$. We will call $H^{2,1}_a=H^{2,1,\alpha}_{+,\sigma,+,\sigma}+\Delta^{2,1,\beta}_{+,\sigma,+,\sigma}$, and

$$|H^{2,1}_a(\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2)|\leqslant\bar{g}_h\frac{\gamma^{-2h}}{Z_h}. \tag{6.15}$$

By the phase transformation $\psi^\pm_{+,-\sigma}\to e^{\pm\alpha_{\mathbf{x}}}\psi^\pm_{+,-\sigma}$, $\psi^\pm_{+,\sigma}\to\psi^\pm_{+,\sigma}$ and $\psi^\pm_{-,\pm\sigma}\to\psi^\pm_{-,\pm\sigma}$, and using a correction identity similar to (6.13) we find

$$\begin{aligned}&-\nu^b_{+,\sigma}D_+(\mathbf{p})G^{2,1}_{+,\sigma,+,\sigma}+(1-\nu^b_{+,-\sigma})D_+(\mathbf{p})G^{2,1}_{+,\sigma,+,-\sigma}\\ &-\nu^b_{-,\sigma}D_-(\mathbf{p})G^{2,1}_{+,\sigma,-,\sigma}-\nu^b_{-,-\sigma}D_-(\mathbf{p})G^{2,1}_{+,\sigma,-,-\sigma}=H^{2,1}_b\end{aligned} \tag{6.16}$$

where $H^{2,1,a}_b$ verifies a bound similar to (6.15).

In the same way by the Gauge transformation $\psi^{\pm}_{-,\sigma} \to e^{\pm\alpha_{\mathbf{x}}}\psi^{\pm}_{-,\sigma}$, $\psi^{\pm}_{-,-\sigma} \to \psi^{\pm}_{-,-\sigma}$ and $\psi^{\pm}_{+,\pm\sigma} \to \psi^{\pm}_{+,\pm\sigma}$, and using a correction identity similar to (6.13) we find

$$\begin{aligned}&- \nu^c_{+,\sigma} D_+(\mathbf{p}) G^{2,1}_{+,\sigma,+,\sigma} - \nu^c_{+,\sigma} D_+(\mathbf{p}) G^{2,1}_{+,\sigma,+,-\sigma} + (1-\nu^c_{-,\sigma}) D_-(\mathbf{p}) G^{2,1}_{+,\sigma,-,\sigma}\\ &- \nu^c_{-,-\sigma} D_-(\mathbf{p}) G^{2,1}_{+,\sigma,-,-\sigma} = H^{2,1}_c \end{aligned} \tag{6.17}$$

where $H^{2,1,a}_c$ verifies a bound similar to (6.15).

Finally by the Gauge transformation $\psi^{\pm}_{-,-\sigma} \to e^{\pm\alpha_{\mathbf{x}}}\psi^{\pm}_{-,-\sigma}$, $\psi^{\pm}_{-,\sigma} \to \psi^{\pm}_{-,\sigma}$ and $\psi^{\pm}_{+,\pm\sigma} \to \psi^{\pm}_{+,\pm\sigma}$ we get the following WI

$$\begin{aligned}&- \nu^d_{+,\sigma} D_+(\mathbf{p}) G^{2,1}_{+,\sigma,+.\sigma} - \nu^d_{+,-\sigma} D_+(\mathbf{p}) G^{2,1}_{+,\sigma,+,-\sigma}\\ &- \nu^d_{-,\sigma} D_-(\mathbf{p}) G^{2,1}_{+,\sigma,-,\sigma} + (1-\nu^d_{-,-\sigma}) D_-(\mathbf{p}) G^{2,1}_{+,\sigma,-,-\sigma} = H^{2,1}_d \end{aligned} \tag{6.18}$$

Bounds like (6.15) and (6.14) hold also for $H^{2,1}_b, H^{2,1}_c, H^{2,1}_d$. It is easy to see from some algebra that the above relations imply

$$\begin{aligned}&D_+(\mathbf{p}) G^{2,1}_{+,\sigma,+,\sigma}(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)\\ &\quad = G^2_+(\mathbf{k}_1) - G^2_+(\mathbf{k}_2) + (1+F^1_a) H^{2,1}_a + F^1_b H^{2,1}_b + F^1_c H^{2,1}_c + F^1_d H^{2,1}_d \end{aligned} \tag{6.19}$$

$$\begin{aligned}D_+(\mathbf{p}) G^{2,1}_{+,\sigma,+,-\sigma}(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2) &= (1+F^2_a) H^{2,a}_a + F^2_b H^{2,1}_b\\ &\quad + F^2_c H^{2,1}_c + F^2_d H^{2,1}_d \end{aligned} \tag{6.20}$$

$$\begin{aligned}D_-(\mathbf{p}) G^{2,1}_{+,\sigma,-,\sigma}(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2) &= (1+F^3_a) H^{2,a}_a + F^3_b H^{2,1}_b\\ &\quad + F^3_c H^{2,1}_c + F^3_d H^{2,1}_d \end{aligned} \tag{6.21}$$

$$\begin{aligned}D_-(\mathbf{p}) G^{2,1}_{+,\sigma,-,-\sigma}(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2) &= (1+F^4_a) H^{2,a}_a + F^4_b H^{2,1}_b\\ &\quad + F^4_c H^{2,1}_c + F^4_d H^{2,1}_d \end{aligned} \tag{6.22}$$

with F_a, F_b, F_c, F_d are combinations of the ν, with the property that if $|\nu^j_i| \leqslant C\bar{g}$, then $|F_i| \leqslant C\bar{g}_h$. Then, (6.19) really provides a relation between $G^{2,1}$ and G^2 up to bounded corrections.

6.2. Proof of Lemma 5

We introduce the generating function for $H_a^{2,1}$

$$\int P_L(d\psi)e^{-\mathcal{V}_L+T_1}\exp\Big[\int d\mathbf{k}d\mathbf{p}J_\mathbf{p}\nu^a_{+,\sigma}D_+(\mathbf{p})\psi^+_{\mathbf{k},+,\sigma}\psi^-_{\mathbf{k}+\mathbf{p},+\sigma}$$
$$+\int d\mathbf{k}d\mathbf{p}J_\mathbf{p}[\nu^a_{+,-\sigma}D_+(\mathbf{p})\psi^+_{\mathbf{k},+,-\sigma}\psi^-_{\mathbf{k}+\mathbf{p},+,-\sigma}$$
$$+\nu^a_{-,\sigma}D_-(\mathbf{p})\psi^+_{\mathbf{k},-,\sigma}\psi^-_{\mathbf{k}+\mathbf{p},-,\sigma}+\nu^a_{-,-\sigma}D_-(\mathbf{p})\psi^+_{\mathbf{k},-,-\sigma}\psi^-_{\mathbf{k}+\mathbf{p},-,-\sigma}\Big] \tag{6.23}$$

where

$$T_1=\int d\mathbf{k}d\mathbf{p}J_\mathbf{p}C(\mathbf{k},\mathbf{k}+\mathbf{p})\psi^+_{\mathbf{k},+,\sigma}\psi^-_{\mathbf{k}+\mathbf{p},+,\sigma} \tag{6.24}$$

The analysis proceeds essentially identical to the one of Section 4 of ref. 5. After integrating the ψ^0 field, we get in the effective potential a sum of monomials of the form $WJ^m\psi_1\dots\psi_n$; we extend the definition of $\mathcal{L}$ to monomials of this kind by requiring that it acts non trivially only on the terms linear in J and quadratic in ψ, as a power counting argument shows that they are the only marginal terms.

Consider now the terms in which T_1 is contracted; they are of the form

$$\sum_{\tilde\omega,\tilde\sigma}\int d\mathbf{p}\int d\mathbf{k}^+J_\mathbf{p}\hat\psi^+_{\mathbf{k}^+,\tilde\omega,\tilde\sigma}\hat\psi^-_{\mathbf{k}^+-\mathbf{p},\tilde\omega,\tilde\sigma}[F^{(-1)}_{2,+,\sigma,\tilde\omega,\tilde\sigma}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p})$$
$$+F^{(-1)}_{1,+,\sigma}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p})\delta_{+,\tilde\omega}\delta_{\sigma,\tilde\sigma}] \tag{6.25}$$

where $F^{(-1)}_{2,+,\sigma,\tilde\omega,\tilde\sigma}$ is given by all the terms obtained contracting both the ψ fields in T_1 while $F^{(-1)}_{1,+,\sigma}$ are is given by the terms obtained leaving external one of the ψ-fields of T_1. Both contributions to the r.h.s. of (3.39) are dimensionally marginal; however, the renormalization of $F^{(-1)}_{1,\omega,\sigma}$ is trivial, as it is of the form

$$F^{(-1)}_{1,+,\sigma}(\mathbf{k}^+,\mathbf{k}^-)=\frac{[C_{h,0}(\mathbf{k}^-)-1]D_+(\mathbf{k}^-)\hat g^{(0)}_\omega(\mathbf{k}^+)-u_0(\mathbf{k}^+)}{D_+(\mathbf{k}^+-\mathbf{k}^-)}G^{(2)}(\mathbf{k}^+) \tag{6.26}$$

or the similar one, obtained exchanging $\mathbf{k}^+$ with $\mathbf{k}^-$.

By the oddness of the propagator in the momentum, $G^{(2)}(0)=0$, hence we can regularize such term without introducing any local term, by simply rewriting it as

$$F^{(-1)}_{1,+,\sigma}(\mathbf{k}^+,\mathbf{k}^-)=\frac{[C_{h,0}(\mathbf{k}^-)-1]D_\omega(\mathbf{k}^-)\hat{g}^{(0)}_+(\mathbf{k}^+)-u_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+-\mathbf{k}^-)}\times[G^{(2)}(\mathbf{k}^+)-G^{(2)}(0)]. \qquad (6.27)$$

As shown in ref. 5, by using the symmetry property

$$\hat{g}^{(j)}_\omega(\mathbf{k})=-i\omega\hat{g}^{(j)}_\omega(\mathbf{k}^*),\quad \mathbf{k}=(k,k_0),\quad \mathbf{k}^*=(-k_0,k), \qquad (6.28)$$

$F^{(-1)}_{2,\omega,\sigma,\tilde{\omega},\tilde{\sigma}}$ can be written as

$$F^{-1}_{2,\omega,\sigma,\tilde{\omega},\tilde{\sigma}}(\mathbf{k}^+,\mathbf{k}^-)=\frac{1}{D_\omega(\mathbf{p})}\left[p_0A_{0,\omega,\sigma,\tilde{\omega},\tilde{\sigma}}(\mathbf{k}^+,\mathbf{k}^-)+p_1A_{1,\omega,\sigma,\tilde{\omega},\tilde{\sigma}}(\mathbf{k}^+,\mathbf{k}^-)\right], \qquad (6.29)$$

where $A_{i,\omega,\sigma,\tilde{\omega},\tilde{\sigma}}(\mathbf{k}^+,\mathbf{k}^-)$ are functions such that, if we define

$$\mathcal{L}F^{-1}_{2,+,\sigma,\tilde{\omega},\pm\sigma}=\frac{1}{D_+(\mathbf{p})}\left[p_0A_{0,+,\sigma,\tilde{\omega},\pm\sigma}(0,0)+p_1A_{1,+,\sigma,\tilde{\omega},\pm\sigma}(0,0)\right], \qquad (6.30)$$

then,

$$\mathcal{L}F^{-1}_{2,+,\sigma,\tilde{\omega},\pm\sigma}=D_{\tilde{\omega}}(\mathbf{p})Z^{3,\tilde{\omega},\pm\sigma}_{-1}, \qquad (6.31)$$

where $Z^{3,\tilde{\omega},\pm\sigma}_{-1}$ are four suitable real constants.

Consider now the terms in which the $\nu_{\omega,\sigma}$ are contracted; we define the localization operator on such terms as

$$\mathcal{L}\int d\mathbf{k}d\mathbf{p}D_\omega(\mathbf{p})W^{-1}_{\omega,\pm\sigma}(\mathbf{k},\mathbf{k}-\mathbf{p})\psi^+_{\mathbf{k},\omega,\pm\sigma}\psi^-_{\mathbf{k}+\mathbf{p},\omega,\pm\sigma}=\int d\mathbf{k}d\mathbf{p}D_\omega(\mathbf{p})W^{-1}_{\omega,\pm\sigma}(\mathbf{0},\mathbf{0})\psi^+_{\mathbf{k},\omega,\pm\sigma}\psi^-_{\mathbf{k}+\mathbf{p},\omega,\pm\sigma} \qquad (6.32)$$

We define $\nu_{-1,\omega,\pm\sigma} = Z_{-1}^{3,\omega,\pm\sigma} + W_{\omega,\pm\sigma}^{-1}$ we get that the local terms linear in J are

$$\int d\mathbf{k}d\mathbf{p}J_{\mathbf{p}}\Big[\nu_{-1,+,\sigma}^{a}D_{+}(\mathbf{p})\psi_{\mathbf{k},+,\sigma}^{+}\psi_{\mathbf{k+p},+\sigma}^{-}+\nu_{-1,+,-\sigma}^{a}D_{+}(\mathbf{p})\psi_{\mathbf{k},+,-\sigma}^{+}\psi_{\mathbf{k+p},+,-\sigma}^{-} \\ +\nu_{-1,-,\sigma}^{a}D_{-}(\mathbf{p})\psi_{\mathbf{k},-,\sigma}^{+}\psi_{\mathbf{k+p},-,\sigma}^{-}+\nu_{-1,-,-\sigma}^{a}D_{-}(\mathbf{p})\psi_{\mathbf{k},-,-\sigma}^{+}\psi_{\mathbf{k+p},-,-\sigma}^{-}\Big] \tag{6.33}$$

We can iterate the above procedure; at the integration of the generic scale the terms quadratic and linear in J in the effective potential are obtained contracted a T_1 vertex (in such a case one of the two fields of T_1 is necessarily contracted at scale 0) or a $\nu_{k,\omega,\sigma}$ vertex; in both case the preceding analysis can be repeated and the local terms linear in J are, for $k > h$

$$\int d\mathbf{k}d\mathbf{p}J_{\mathbf{p}}[\nu_{k,+,\sigma}^{a}D_{+}(\mathbf{p})\psi_{\mathbf{k},+,\sigma}^{+}\psi_{\mathbf{k+p},+\sigma}^{-}+\nu_{k,+,-\sigma}^{a}D_{+}(\mathbf{p})\psi_{\mathbf{k},+,-\sigma}^{+}\psi_{\mathbf{k+p},+,-\sigma}^{-} \\ +\nu_{k,-,\sigma}^{a}D_{-}(\mathbf{p})\psi_{\mathbf{k},-,\sigma}^{+}\psi_{\mathbf{k+p},-,\sigma}^{-}+\nu_{k,-,-\sigma}^{a}D_{-}(\mathbf{p})\psi_{\mathbf{k},-,-\sigma}^{+}\psi_{\mathbf{k+p},-,-\sigma}^{-}] \tag{6.34}$$

We have then obtained an expansion for $H_a^{2,1}$ in which new running coupling constants appear, namely $\nu_{k,\omega,\pm\sigma}$; the analogue of Theorem 3 ensures convergence $\nu_{k,\omega,\pm\sigma}$ are small or any $k > h$. The beta function for $\nu_{k,\omega,\pm\sigma}$ has the following form

$$\nu_{k-1,\omega,\pm\sigma} = \nu_{k,+,\pm\sigma} + \beta_{\omega,\pm\sigma}^{1,k}(v_k^L, \dots, v_0^L) + \beta_{\omega,\pm\sigma}^{2,k}(v_k^L, \nu_k, \dots, v_0^L, \nu_0) \tag{6.35}$$

where by definition $\beta_{\omega,\pm\sigma}^{2,k}(v_k^L, \nu_k, \dots, v_0^L, \nu_0)$ is obtained contracting a ν_j while $\beta_{\omega,\pm\sigma}^{1,k}(v_k^L, \dots, v_0^L)$ is obtained contracting T_1 and

$$\left|\beta_{\omega,\pm\sigma}^{1,k}(v_k^L, \dots, v_0^L)\right| \leqslant C\bar{g}_k\gamma^{\theta k}$$

for some constant $0 < \theta < 1$. The presence of the factor $\gamma^{\theta k}$ in the above bound is due to the fact that, for the support properties of the function $C(\mathbf{k}^+, \mathbf{k}^-)$ discussed after (6.24), one of the fields of T_1 is necessarily contracted at scale 0.

In fact we can show (proceeding as in the proof of Lemma 3, or in Section 4.6 of ref. 6) that there exists a sequence ν_k such that $|\nu_{k,\omega,\pm\sigma}| \leqslant C\bar{g}_h\gamma^{\theta k}$ by solving

$$\nu_{k,\omega,\pm\sigma} = -\sum_{k'=h+1}^{k}\left\{\beta_{\omega,\pm\sigma}^{1,k'}(v_k^L, \dots, v_0^L) + \beta_{\omega,\pm\sigma}^{2,k'}(v_k^L, \nu_k, \dots, v_0^L, \nu_0)\right\} \tag{6.36}$$

This shows that there exist $\nu_{\omega,\pm\sigma}$ such that $\nu_{k,\omega,\pm\sigma} = O(\gamma^{\theta k})$.

We have then find an expansion for $H_a^{2,1}(\bar{\mathbf{k}}_1 - \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2)$ very similar to the one of $G^{2,1}$, but in which each tree contributing to $H_a^{2,1}(\bar{\mathbf{k}}_1 - \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2)$ have an extra $\gamma^{\theta h}$; in fact or there is an endpoint ν_k (and we use that $\nu_{k,\omega,\pm\sigma} = O(\gamma^{\theta k})$ and the fact that, as the dimension are negative, the value of the tree has an extra $\gamma^{\theta(h-k)}$) or there is an endpoint T_1 contracted at scale 0 (hence, as the dimensions are negative, the value of the tree has an extra $\gamma^{\theta h}$). ∎

Inserting (6.20) in the Dyson equation, and using (6.15) and (6.14), we see that the first three addenda of the Dyson equation are given by $\left(g_2^0 + O(\bar{g}_h^2)\right) \frac{\gamma^{-4h}}{Z_h^2}$.

We have to consider now the last three addenda in the Dyson equation; let us start by

$$\int d\mathbf{p} \left[g_2^o \langle \psi^+_{\bar{\mathbf{k}}_1,+,\sigma} \psi^-_{\bar{\mathbf{k}}_2,+,\sigma} \psi^+_{\bar{\mathbf{k}}_3,-,-\sigma} \psi^-_{\bar{\mathbf{k}}_4-\mathbf{p},-,-\sigma} \rho_{\mathbf{p},+,\sigma} \rangle_T \right] \tag{6.37}$$

Let us call

$$G^{4,1}_{+,\sigma,\omega'\sigma'}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}; \mathbf{p}) = \langle \psi^+_{\mathbf{k}_1,+,\sigma} \psi^-_{\mathbf{k}_2,+,\sigma} \psi^+_{\mathbf{k}_3,-,-\sigma} \psi^-_{\mathbf{k}_4-\mathbf{p},-,-\sigma} \rho_{\mathbf{p},\omega',\sigma'} \rangle_T \tag{6.38}$$

As $|\bar{\mathbf{k}}_4| = \gamma^h$ the support properties of the propagators imply that $|\mathbf{p}| \leqslant \gamma + \gamma^h \leqslant 2\gamma$, hence we can freely multiply $G_+^{4,1}$ in the r.h.s. of (6.37) by the compact support function $\chi_0(\gamma^{-j_m}|\mathbf{p}|)$, with $j_m = [1 + \log_\gamma 2] + 1$. It follows that (6.37) can be written as

$$\int d\mathbf{p} \chi_M(\mathbf{p}) G_+^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) + \int d\mathbf{p} \tilde{\chi}_M(\mathbf{p}) G_+^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) \tag{6.39}$$

where $\chi_M(\mathbf{p})$ is a compact support function vanishing for $|\mathbf{p}| \geqslant \gamma^{h+j_m-1}$ and

$$\tilde{\chi}_M(\mathbf{p}) = \sum_{h_p = h + j_m}^{j_m} f_{h_p}(\mathbf{p}). \tag{6.40}$$

Note that the decomposition of the $\mathbf{p}$ sum is done so that $\tilde{\chi}_M(\mathbf{p}) = 0$ if $|\mathbf{p}| \leqslant 2\gamma^h$. It is easy to show that the first term in (6.39) is bounded by $O\left(\frac{\bar{g}_h^2\gamma^{-3h}}{Z_h^2}\right)$, see ref. 6. Regarding the second addend we will use the following Ward identities

$$\begin{aligned}&(1-\nu^a_{+,\sigma})D_+(\mathbf{p})G^{4,1}_{+,\sigma,+,\sigma}-\nu^a_{+,-\sigma}D_+(\mathbf{p})G^{4,1}_{+,\sigma,+,-\sigma}-\nu^a_{-,\sigma}D_-(\mathbf{p})G^{4,1}_{+,\sigma,-,\sigma}\\&\quad-\nu^a_{-,-\sigma}D_-(\mathbf{p})G^{4,1}_{+,\sigma,-,-\sigma}=G^4_+(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})\\&\quad-G^4_+(\mathbf{k}_1,\mathbf{k}_2+\mathbf{p},\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})+H^{4,1}_a(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p};\mathbf{p})\end{aligned}\tag{6.41}$$

$$\begin{aligned}&-\nu^b_{+,\sigma}D_+(\mathbf{p})G^{4,1}_{+,\sigma,+,\sigma}+(1-\nu^b_{+,-\sigma})D_+(\mathbf{p})G^{4,1}_{+,\sigma,+,-\sigma}-\nu^b_{-,\sigma}D_-(\mathbf{p})G^{2,1}_{+,\sigma,-,\sigma}\\&\quad-\nu^b_{-,-\sigma}D_-(\mathbf{p})G^{4,1}_{+,\sigma,-,-\sigma}=H^{4,1}_b\end{aligned}\tag{6.42}$$

$$\begin{aligned}&-\nu^c_{+,\sigma}D_+(\mathbf{p})G^{4,1}_{+,\sigma,+.\sigma}-\nu^c_{+,-\sigma}D_+(\mathbf{p})G^{4,1}_{+,\sigma,+,-\sigma}+(1-\nu^c_{-,\sigma})D_-(\mathbf{p})G^{4,1}_{+,\sigma,-,\sigma}\\&\quad-\nu^c_{-,-\sigma}D_-(\mathbf{p})G^{4,1}_{+,\sigma,-,-\sigma}=H^{4,1}_c\end{aligned}\tag{6.43}$$

$$\begin{aligned}&-\nu^d_{+,\sigma}D_+(\mathbf{p})G^{4,1}_{+,\sigma,+.\sigma}-\nu^d_{+,-\sigma}D_+(\mathbf{p})G^{4,1}_{+,\sigma,+,-\sigma}-\nu^d_{-,\sigma}D_-(\mathbf{p})G^{2,1}_{+,\sigma,-,\sigma}\\&\quad+(1-\nu^d_{-,-\sigma})D_-(\mathbf{p})G^{4,1}_{+,\sigma,-,-\sigma}=G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3-\mathbf{p},\mathbf{k}_4-\mathbf{p})\\&\quad-G^4(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)+H^{2,1}_d(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p};\mathbf{p})\end{aligned}\tag{6.44}$$

where the functions $H^{4,1}_i$ are defined in an analogous way to the functions $H^{2,1}_i$. It is easy to see from some algebra that the above relations imply

$$\begin{aligned}&D_+(\mathbf{p})G^{4,1}_{+,\sigma,-,-\sigma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p};\mathbf{p})\\&\quad=(1+G^a_1)[G^4_+(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}-\mathbf{p})-G^4_+(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}-\mathbf{p})]\\&\qquad+G^a_2[G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3-\mathbf{p},\mathbf{k}_4-\mathbf{p})-G^4(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)]\\&\qquad+(1+G^a_3)H^{4,a}_a+G^a_4H^{4,1}_b+G^5_aH^{4,1}_c+G^6_aH^{4,1}_d\end{aligned}\tag{6.45}$$

with $G^i_a=O(\bar{g})$. From (6.19) we can decompose $\int d\mathbf{p}\tilde{\chi}_M(\mathbf{p})G^{4,1}_+$ as sum of several terms; the one involving $G^4(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ is vanishing while the other three terms involving the other functions G^4 have a bound $O\left(\frac{\bar{g}_h^2\gamma^{-3h}}{Z_h^2}\right)$, see ref. 6. Finally the following results holds

Lemma 6. If the functions $\nu_{\omega,\pm\sigma}$ are the same as in Lemma 7, it holds that, for $i=a,b,c,d$

$$\left|\int d\mathbf{p} g_-(\mathbf{k}_4)\frac{H_i^{4,1}}{D_+(\mathbf{p})}\right| \leqslant C\frac{\bar{g}_h^2\gamma^{-3h}}{Z_h^2} \tag{6.46}$$

Inserting all the above bounds in the Dyson equation (4.1) computed at momenta $|\mathbf{k}_i|=\gamma^h, i=1,2,3,4$ we have completed the proof of Lemma 3.

Lemma 6 is proved considering

$$\tilde{G}_+^4(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=\frac{\partial^4}{\partial\phi^+_{\mathbf{k}_1,+,\sigma}\partial\phi^-_{\mathbf{k}_2,+,\sigma}\partial\phi^+_{\mathbf{k}_3,-,-\sigma}\partial J_{\mathbf{k}_4}}\tilde{W}\Bigg|_{\phi=0}, \tag{6.47}$$

where

$$\tilde{W}=\log\int P(d\hat{\psi})e^{-T(\psi)+\nu_1^a T_1(\psi)+\nu_2^a T_2(\psi)+\nu_3^a T_3(\psi)+\nu_4^a T_4(\psi)+\sum_\omega\int d\mathbf{x}[\phi^+_{\mathbf{x},\omega}\hat{\psi}^-_{\mathbf{x},\omega}+\hat{\psi}^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}]}, \tag{6.48}$$

$$T(\psi)=\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{C_+(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_+(\mathbf{p})}(\hat{\psi}^+_{\mathbf{k},+,\sigma}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},+,\sigma})\hat{\psi}^+_{\mathbf{k}_4-\mathbf{p},-,-\sigma}\hat{J}_{\mathbf{k}_4}\hat{g}_-(\mathbf{k}_4), \tag{6.49}$$

$$T_1(\psi)=\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}(\hat{\psi}^+_{\mathbf{k},+,\sigma}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},+,\sigma})\hat{\psi}^+_{\mathbf{k}_4-\mathbf{p},-,-\sigma}\hat{J}_{\mathbf{k}_4}\hat{g}_-(\mathbf{k}_4), \tag{6.50}$$

$$T_2(\psi)=\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}(\hat{\psi}^+_{\mathbf{k},+,-\sigma}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},+,-\sigma})\hat{\psi}^+_{\mathbf{k}_4-\mathbf{p},-,-\sigma}\hat{J}_{\mathbf{k}_4}\hat{g}_-(\mathbf{k}_4). \tag{6.51}$$

$$T_3(\psi)=\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}(\hat{\psi}^+_{\mathbf{k},-,\sigma}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},-,\sigma})\hat{\psi}^+_{\mathbf{k}_4-\mathbf{p},-,-\sigma}\hat{J}_{\mathbf{k}_4}\hat{g}_-(\mathbf{k}_4). \tag{6.52}$$

$$T_4(\psi)=\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}(\hat{\psi}^+_{\mathbf{k},-,-\sigma}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},-,-\sigma})\hat{\psi}^+_{\mathbf{k}_4-\mathbf{p},-,-\sigma}\hat{J}_{\mathbf{k}_4}\hat{g}_-(\mathbf{k}_4). \tag{6.53}$$

It holds that

$$\tilde{G}^4_+(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4) = \int d\mathbf{p} g_-(\mathbf{k}_4) \frac{H_i^{4,1}}{D_+(\mathbf{p})} \tag{6.54}$$

Note that the expansion of $\tilde{G}^4_+$ is very similar to the expansion of G^4_+, except for the presence of a special vertex associated to J. The proof of the bound (6.46) is essentially identical to the one for the spinless case of ref. 6, to which we refer for the technical details.

7. CORRELATION FUNCTIONS

Once that the multiscale analysis of the partition function is completed, it is possible to apply the same ideas and methods to the Grassmann integrals giving the Schwinger function or the correlations; as the analysis is essentially identical to the one in ref. 4, we will give only the main ideas referring to Section 5 of ref. 4 for details. The density-density correlation can be written in terms of a Grassmann integral in the following way

$$\langle \rho(\mathbf{x})\rho(\mathbf{y})\rangle_T = \langle \rho(\mathbf{x})\rho(\mathbf{y})\rangle - \langle \rho(\mathbf{x})\rangle \langle \rho(\mathbf{y})\rangle = \frac{\partial^2 \mathcal{S}}{\partial\phi(\mathbf{x})\partial\phi(\mathbf{y})} \tag{7.1}$$

where

$$\mathcal{S}(\phi) = \log \int P(d\psi) e^{-\mathcal{V} - \sum_\sigma \int d\mathbf{x} \phi(\mathbf{x}) \psi^+_{\mathbf{x},\sigma} \psi^-_{\mathbf{x},\sigma}} \tag{7.2}$$

We shall evaluate $\mathcal{S}$ in a way which is very close to that used for the integration of the partition function in Section 2. We introduce the scale decomposition described above and we perform iteratively the integration of the single scale fields, starting from the field of scale 1.

After integrating the fields $\psi^{(1)}, \dots, \psi^{(h+1)}$ we find

$$e^{\mathcal{S}(\phi)} = e^{-L\beta E_h + S^{(h+1)}(\phi)} \int P_{Z_h, C_h}(d\psi^{\leqslant h}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}, \phi)}, \tag{7.3}$$

where $P_{Z_h,\sigma_h,C_h}(d\psi^{(\leqslant h)})$ and $\mathcal{V}^h$ are given by (3.15) and (3.16), respectively, while $S^{(h+1)}$ (ϕ), which denotes the sum over all the terms dependent on ϕ but independent of the ψ field, and $\mathcal{B}^{(h)}(\psi^{(\leqslant h)}, \phi)$, which denotes the sum over all the terms containing at least one ϕ field and two ψ fields, can be represented in the form, if $\int d\mathbf{x} = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x\in\Lambda}$

$$S^{(h+1)}(\phi) = \sum_{m=1}^{\infty} \int d\mathbf{x}_1 \cdots d\mathbf{x}_m S_m^{(h+1)}(\mathbf{x}_1, \dots, \mathbf{x}_m) \left[\prod_{i=1}^{m} \phi(\mathbf{x}_i)\right] \tag{7.4}$$

$$\mathcal{B}^{(h)}\left(\psi^{(\leqslant h)}, \phi\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\underline{\sigma},\underline{\omega}} \int d\mathbf{x}_1 \cdots d\mathbf{x}_m d\mathbf{y}_1 \cdots d\mathbf{y}_{2n}$$
$$\cdot B^{(h)}_{m,2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1, \dots, \mathbf{x}_m; \mathbf{y}_1, \dots, \mathbf{y}_{2n}) \left[\prod_{i=1}^{m} \phi(\mathbf{x}_i)\right]\left[\prod_{i=1}^{2n} \psi^{(\leqslant h)\sigma_i}_{\mathbf{y}_i,\omega_i}\right]. \tag{7.5}$$

Since the field ϕ is equivalent, from the point of view of dimensional considerations, to two ψ fields, the only terms in the r.h.s. of (7.5) which are not irrelevant are those with $m=1$ and $n=1$, which are marginal. Hence we extend the definition of the localization operator $\mathcal{L}$, so that its action on $\mathcal{B}^{(h)}\left(\psi^{(\leqslant h)}, \phi\right)$ is described in the following way, by its action on the kernels $B^{(h)}_{m,2n,\sigma,\omega}(\mathbf{p}, \mathbf{k}_1, \dots, \mathbf{k}_n)$:

(1) if $m=1$, $n=1$ then

$$\mathcal{L}B^{(h)}_{1,2,\sigma,\omega}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2) = B^{(h)}_{1,2,\sigma,\omega}(0; 0, 0) \tag{7.6}$$

(2) $\mathcal{L}=0$ in all the other cases

It follows that

$$\mathcal{L}\mathcal{B}^{(h)}\left(\psi^{(\leqslant h)}, \phi\right) = \frac{Z_h^{(1)}}{Z_h} F_1^{(\leqslant h)} + \frac{Z_h^{(2)}}{Z_h} F_2^{(\leqslant h)}, \tag{7.7}$$

where $Z_h^{(1)}$ and $Z_h^{(2)}$ are real numbers, such that $Z_1^{(1)} = Z_1^{(2)} = 1$ and

$$F_1^{(\leqslant h)} = \sum_{\omega,\sigma} \int d\mathbf{x} \phi(\mathbf{x}) e^{2i\omega \mathbf{p}_F x} \psi^{(\leqslant h)+}_{\mathbf{x},\omega,\sigma} \psi^{(\leqslant h)-}_{\mathbf{x},-\omega,\sigma}, \tag{7.8}$$

$$F_2^{(\leqslant h)} = \sum_{\sigma=\pm 1} \int d\mathbf{x} \phi(\mathbf{x}) \psi^{(\leqslant h)\sigma}_{\mathbf{x},\omega,\sigma} \psi^{(\leqslant h)-}_{\mathbf{x},\omega,\sigma}. \tag{7.9}$$

By using the notation of Section 2, we can write the integral in the r.h.s. of (7.3) as

$$e^{-L\beta t_h}\int P_{\tilde{Z}_{h-1},C_h}(\mathrm{d}\psi^{(\leqslant h)})e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)})+\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)},\phi)}$$
$$=e^{-L\beta t_h}\int P_{Z_{h-1},C_{h-1}}(d\psi^{(\leqslant h-1)})$$
$$\times\int P_{Z_{h-1},\tilde{f}_h^{-1}}(\mathrm{d}\psi^{(h)})e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)},\phi)}, \tag{7.10}$$

where $\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)})$ is defined as in Section 3 and

$$\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)},\phi)=\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)},\phi). \tag{7.11}$$

$\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h-1)},\phi)$ and $S^{(h)}(\phi)$ are then defined through

$$e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h-1)})+\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h-1)},\phi)-L\beta\tilde{E}_h+\tilde{S}^{(h)}(\phi)}$$
$$=\int P_{Z_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)},\phi)}. \tag{7.12}$$

Of course also the new renormalization constants related to the density-density correlation function obey to a Beta function equation of the form

$$\frac{Z_{h-1}^{(i)}}{Z_h^{(i)}}=1+z_h^{(i)},\quad i=1,2, \tag{7.13}$$

where $z_h^{(1)}$ and $z_h^{(2)}$ are some quantities of order $\bar{g}_h$. It turns out that $\lim_{h\to-\infty}\frac{Z_h^1}{\gamma^{\eta_1 h}}=1+O(U)$ while $\lim_{h\to-\infty}Z_h^1=1+O(U)$, with $\eta_1=-bU+O(U^2)$ and $b>0$ is a suitable constant. The bounds for the expansion of the Schwinger function or the correlation functions are done exactly as in Section 5 of ref. 4; to the first term in (1.9) or to the first two terms in (1.12) contribute only trees with only endpoints with scale $\leqslant 0$; the other trees have at least an endpoint at scale 1 so that by the short memory property they have a faster decay.

8. THE HUBBARD MODEL IN A MAGNETIC FIELD

We only sketch the analysis when there is a magnetic field as it is indeed very similar to analysis of the vanishing magnetic field case.

The presence of a magnetic field destroys the $SU(2)$ spin symmetry. The counterterms are introduced by the following definition

$$\tilde{t}=t-\sum_{\sigma}\delta_{\sigma} \quad \cos p_F^{\sigma}=\mu+\operatorname{sign}(\sigma)h-\nu_{\sigma} \tag{8.1}$$

This means that $\mathcal{V}$ in the partition function (2.5) is replaced by

$$\mathcal{V}=U\int_{-\beta/2}^{\beta/2}\mathrm{d}x_0\sum_{x}\psi^{+}_{\mathbf{x},+}\psi^{-}_{\mathbf{x},+}\psi^{+}_{\mathbf{x},-}\psi^{-}_{\mathbf{x},-}+\int_{-\beta/2}^{\beta/2}\mathrm{d}x_0\sum_{x,\sigma}\nu_{\sigma}\psi^{+}_{\mathbf{x},\sigma}\psi^{-}_{\mathbf{x},\sigma}$$
$$+\int_{-\beta/2}^{\beta/2}dx_0\sum_{x,y,\sigma}\delta_{\sigma}t_{x,y}\psi^{+}_{\mathbf{x},\sigma}\psi^{-}_{\mathbf{x},\sigma} \tag{8.2}$$

The ultraviolet and infrared integration are done as in §2,§3, with the difference that for $h\leqslant\bar{h}$ only quartic monomials verifying $\|\sum_{i=1}^{4}\varepsilon_i\omega_i p_F^{\sigma_i}\|=0$ (instead of (3.18)) are present in the effective potential. The definition of $\mathcal{L}$ on the quartic terms is similar to (3.19) with the difference that the delta function in the (3.19) is replaced by $\delta(\sum_{i=1}^{4}\varepsilon_i\omega_i p_F^{\sigma_i})$. This means that the quartic marginal terms verify $\sum_{i=1}^{4}\varepsilon_i\omega_i p_F^{\sigma_i}=0$ mod. 2π and this condition forbids the configuration of ω given by the second of (3.22), if h is small enough, as $p_F(\sigma)-p_F(-\sigma)+n\pi\neq 0$; in other words there is no the analogue of the g_1^p-terms in the effective potential. Moreover we are assuming in Theorem 2 that $|\cos^{-1}(\mu+h)+\cos^{-1}(\mu-h)-\pi|\geqslant\bar{C}$ for some constant C; this implies that the configuration $(\omega,-\omega,\omega,-\omega)$ is not allowed.

The wave function renormalization depends by σ, that is $Z_h\equiv Z_{h,\sigma}$, and the relevant part of the effective potential is given by

$$\begin{aligned}
&\mathcal{L}\widehat{\mathcal{V}}^{(h)}\big(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}\big)\\
&\quad=\sum_{\sigma}\big\{\delta_{h,\sigma}F_{a,\sigma}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}+\gamma^h\nu_{h,\sigma}F_{n,\sigma}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)})\\
&\qquad+g^p_{2,h,\sigma}F_{2,\sigma,-\sigma}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}+g^o_{2,h,\sigma}F_{2,\sigma,\sigma}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}\\
&\qquad+g_{4,h,\sigma}F_{4,\sigma,-\sigma}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}
\end{aligned} \tag{8.3}$$

There are then in the $h\neq 0$ case 4 quadratic and 6 quartic running coupling constants; Theorem 3 is still valid if they are small enough.

We can choose $\nu_\sigma, \delta_\sigma$ so that $\nu_{h,+}, \nu_{h,-}, \delta_{h,+}, \delta_{h,-}$ are $O(\bar{g}_h \gamma^{\theta h})$; this is shown by a fixed point argument essentially identical to Lemma 2. The four quartic running coupling g_h^i obeys to equations of the form, if $i = (2o+), (2o-), (2p+), (2p-), (4+), (4-)$

$$g_{h-1}^i = g_h^i + \beta_h^i + R_h^i \tag{8.4}$$

where β_h^i is given by the sum of trees with no endpoints at scale i, only g_L^k propagators and no endpoints to which are associated ν_k, δ_k; if $\nu_{h,+}, \nu_{h,-}, \delta_{h,+}, \delta_{h,-}$ are $O(\bar{g}_h \gamma^{\theta h})$ then, as in Section 5, $R_h^i \leqslant C\bar{g}_h g^{\theta h}$.

The flow of the quartic running constants is even simpler as the one in the $h=0$ case as $|\beta_h^i| \leqslant C\bar{g}_h g^{\theta h}$. This can be proved as in §6 introducing the following reference model, with

$$\begin{aligned}\mathcal{V}_L = \sum_{\omega,\sigma} \int d\mathbf{k}_1 \cdots \int d\mathbf{k}_4 \delta\left(\sum_i \varepsilon_i \mathbf{k}_i\right) \Big[& g_{2,\sigma}^o \psi^+_{\mathbf{k}_1,\omega,\sigma} \psi^-_{\mathbf{k}_2,\omega,\sigma} \psi^+_{\mathbf{k}_3,-\omega,\sigma} \psi^-_{\mathbf{k}_4,-\omega,\sigma} \\ & + g_{2,\sigma}^p \psi^+_{\mathbf{k}_1,\omega,\sigma} \psi^-_{\mathbf{k}_2,\omega,\sigma} \psi^+_{\mathbf{k}_3,-\omega,-\sigma} \psi^-_{\mathbf{k}_4,-\omega,-\sigma} \\ & + g_{4,\sigma} \psi^+_{\mathbf{k}_1,\omega,\sigma} \psi^-_{\mathbf{k}_2,\omega,\sigma} \psi^+_{\mathbf{k}_3,\omega,-\sigma} \psi^-_{\mathbf{k}_4,\omega,-\sigma} \Big]\end{aligned} \tag{8.5}$$

In this reference model the interaction has five independent parameters, instead of three as in the previous case. We can analyze by RG the reference model and we get the couplings $\tilde{g}^p_{h,2,+}, \tilde{g}^p_{h,2,-}, \tilde{g}^o_{h,2,+}, \tilde{g}^o_{h,2,-}, \tilde{g}^o_{h,4,+}, \tilde{g}^o_{h,4,-}$. The proof that their values remains close to the initial value is essentially identical to the analysis in Section 6; (8.6) is replaced by

$$\begin{aligned}&\langle \psi^+_{\mathbf{k}_1,+,\sigma} \psi^-_{\mathbf{k}_2,+,\sigma} \psi^+_{\mathbf{k}_3,-,-\sigma} \psi^-_{\mathbf{k}_4,-,-\sigma} \rangle_T \\ &= g_{-,-\sigma}(\mathbf{k}_4)\{G^2_{-,-\sigma}(\mathbf{k}_3)[g^o_{2,\sigma} \langle \psi^+_{\mathbf{k}_1,+,\sigma} \psi^-_{\mathbf{k}_2,+,\sigma} \rho_{\mathbf{k}_1-\mathbf{k}_2,+,\sigma} \rangle_T \\ &\quad + g^p_{2,-\sigma} \langle \psi^+_{\mathbf{k}_1,+,\sigma} \psi^-_{\mathbf{k}_2,+,\sigma} \rho_{\mathbf{k}_1-\mathbf{k}_2,+,-\sigma} \rangle_T \\ &\quad + g_{4,-\sigma} \langle \psi^+_{\mathbf{k}_1,+,\sigma} \psi^-_{\mathbf{k}_2,+,\sigma} \rho_{\mathbf{k}_1-\mathbf{k}_2,-,-\sigma} \rangle_T] \\ &\quad + \int d\mathbf{p} [g^o_{2,\sigma} \langle \psi^+_{\mathbf{k}_1,+,\sigma} \psi^-_{\mathbf{k}_2,+,\sigma} \psi^+_{\mathbf{k}_3,-,\sigma} \psi^-_{\mathbf{k}_4-\mathbf{p},-,\sigma} \rho_{\mathbf{p},+,\sigma} \rangle_T \\ &\quad + \int d\mathbf{p} g^p_{2,-\sigma} \langle \psi^+_{\mathbf{k}_1,+,\sigma} \psi^-_{\mathbf{k}_2,+,\sigma} \psi^+_{\mathbf{k}_3,-,\sigma} \psi^-_{\mathbf{k}_4-\mathbf{p},-,\sigma} \rho_{\mathbf{p},+,-\sigma} \rangle_T \\ &\quad + \int d\mathbf{p} g_{4,-\sigma} \langle \psi^+_{\mathbf{k}_1,+,\sigma} \psi^-_{\mathbf{k}_2,+,\sigma} \psi^+_{\mathbf{k}_3,+,\sigma} \psi^-_{\mathbf{k}_4-\mathbf{p},+,\sigma} \rho_{\mathbf{p},+,-\sigma} \rangle_T]\}\end{aligned} \tag{8.7}$$

By using the Ward identities of Section 7 one gets that $\tilde{g}^p_{h,2,+}, \tilde{g}^p_{h,2,-}, \tilde{g}^o_{h,2,+}, \tilde{g}^o_{h,2,-}, \tilde{g}^o_{h,4,+}, \tilde{g}^o_{h,4,-}$ remain close to the initial value, and this implies that

β_h^i is asymptotically vanishing. This means that in presence of a magnetic field one has Luttinger liquid behaviour also with a attractive interaction; of course this will be true only if $\bar{h}$ is non vanishing, and it is $O(h^\alpha)$ for some constant $\alpha > 0$ (in fact $\bar{h}$ is finite if $|p_F^\sigma - p_F^{-\sigma}| \neq 0$).

APPENDIX A

A.1. Ultraviolet Decomposition

It is convenient to introduce an ultraviolet cut-off N by writing

$$g^{[1,N]}(\mathbf{x},\mathbf{y}) = \sum_{n=0}^{N} g^{(n)}(\mathbf{x},\mathbf{y}) \tag{A.1}$$

where

$$g^{(n)}(\mathbf{x},\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}} \hat{f}_{u.v.}(\mathbf{k}) h_n(k_0) \frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0 - \tilde{t}\cos k + \tilde{t}\cos p_F} \tag{A.2}$$

with $h_0(k_0) = \chi(k_0)$ and $h_n(k_0) = \chi(\gamma^{-n+1}k_0) - \chi(\gamma^{-n}k_0)$; it holds that $\lim_{N\to\infty} g^{[1,N]}(\mathbf{x},\mathbf{y}) = g^{(u.v.)}(\mathbf{x},\mathbf{y})$ and, for any integer K

$$|g^{(n)}(\mathbf{x},\mathbf{y})| \leqslant \frac{C_K}{1 + (\gamma^n |x_0 - y_0| + |x-y|)^K} \tag{A.3}$$

We define

$$V^{(0)}(\phi) = \lim_{N\to\infty} \log \frac{1}{\mathcal{N}_0} \int P(d\psi^{[1,N]}) e^{\mathcal{V}(\psi^{[1,N]}+\phi)} \tag{A.4}$$

We can integrate iteratively scale by scale, and after the integration of the scales $N, N-1, \ldots, k+1$ we get

$$V^{(k)}(\phi) = \lim_{N\to\infty} \log \frac{1}{\mathcal{N}_k} \int P(d\psi^{[1,k]}) e^{\mathcal{V}(\psi^{[1,k]}+\phi)} \tag{A.5}$$

It is well known that $V^{(k)}$ can be written as sum over *trees* τ similar to the ones in Section 3 (see for instance [15] for the analysis of the ultraviolet problem in the Hubbard model in any dimension) each of them bounded by, if m_v is the number of endpoints of type U following the vertex v on τ

$$C^n [\max(U, |\nu|, |\delta|)]^m \gamma^{-n(m-1)} \prod_v \gamma^{-(n_v - n_{v'})(m_v - 1)} \tag{A.6}$$

One can have $m_v = 1$ only if v is a trivial vertex following the first non trivial vertices on τ; then the terms with $m_v = 1$ correspond to *self-contractions* or *tadpoles*; note however that no divergence are associate to self-contractions as $g^{(n,N)}(\mathbf{x}, \mathbf{x})$ is bounded uniformly in N. Consider then a generic tree with all the sets P_v assigned; the simple expectations over the trivial vertices in the tree with $m_v = 1$ before the first non trivial vertex $\bar{v}$ can be explicitly computed, giving $\psi_{\mathbf{x}}^{+(\leqslant n_{\bar{v}})} \psi_{\mathbf{x}}^{-(\leqslant n_{\bar{v}})} g^{(n_{\bar{v}}, M)}(\mathbf{x}, \mathbf{x})$; the rest of the tree is bounded by an expression like (A.6) with $m_v > 1$, so that by summing over all the scales and the trees the bound (2.17) is found.

A.2. Spin Symmetry

Finally the symmetry property (2.18) follows from the $SU(2)$ invariance of the Hubbard model. A direct way to check this property consists in expanding the truncated expectations corresponding to the integration of $\psi^{u.v.}$ in terms of *Feynmann graphs*. The interaction can be also written in the following way, making more explicit the spin symmetry of the Hubbard model

$$\mathcal{V} = \frac{U}{2} \int dx_0 \sum_x \left(\sum_\sigma \psi^+_{\mathbf{x},\sigma} \psi^-_{\mathbf{x},\sigma} \right) \left(\sum_{\sigma'} \psi^+_{\mathbf{x},\sigma'} \psi^-_{\mathbf{x},\sigma'} \right) + \nu \sum_{x,\sigma} \int dx_0 \psi^+_{\mathbf{x},\sigma} \psi^-_{\mathbf{x},\sigma}$$
$$+ \delta \sum_\sigma \int dx_0 \sum_{x,\sigma} t_{x,y} \psi^+_{\mathbf{x},\sigma} \psi^-_{\mathbf{y},\sigma} \tag{A.7}$$

As usual, the Feymann graphs are obtained representing as vertices the three addenda in (A.7) with four or two oriented half-lines, and contracting in all possible ways the half lines with consistent orientation; it is also convenient to represent the quartic term as a couple of two half-lines connected by a wigghly line, representing the interaction. The value of each Feynmann graph is obtained associating to each line a propagator $g^{u.v}(\mathbf{x}; \mathbf{y})$ and integrating over all the coordinates; the contributions from graphs with four uncontracted half lines has in general the form

$$\int d\mathbf{x}_1 \cdots \int d\mathbf{x}_4 \psi^+_{\mathbf{x}_1,\sigma} \psi^-_{\mathbf{x}_2,\sigma} \psi^+_{\mathbf{x}_3,\sigma'} \psi^+_{\mathbf{x}_4,\sigma'} W^0_{\sigma,\sigma'}(\mathbf{x}_1, \dots, \mathbf{x}_4) \tag{A.8}$$

In order to prove that the kernel is spin-independent, that is

$$W_{\sigma,\sigma} = W_{\sigma,-\sigma} \tag{A.9}$$

we note that in the Feynmann graph we can identify a line of propagators $g^{u,v}(\mathbf{x},\mathbf{y})$ (possibly a point) connecting $\psi^+_{\mathbf{x}_1,\sigma}$ with $\psi^-_{\mathbf{x}_2,\sigma}$, and another line connecting $\psi^+_{\mathbf{x}_3,\sigma'}$ with $\psi^-_{\mathbf{x}_3,\sigma'}$; on such two lines there are points to which are attached wiggly lines to which are attached the fields $\sum_{\sigma''}\psi^+_{\sigma''}\psi^-_{\sigma''}$; the crucial point is that such expression does not depend from the fact that it is connected by the wigghly line to a σ or σ' line. Hence, the contributions to $W^{(0)}_{\sigma,\sigma}$ and $W^{(0)}_{\sigma,-\sigma}$ can possibly differ only because in one case there is a line of propagators σ and in the other case $-\sigma$; but the propagators are spin-independent hence the values of such two contributions are identical (and independent from σ). The same argument can be repeated to prove that $W^{(h)}_{\sigma,\sigma}=W^{(h)}_{\sigma,-\sigma}$, by performing a single scale integration with propagator $g^{(\geqslant h)}(\mathbf{x},\mathbf{y})$.

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